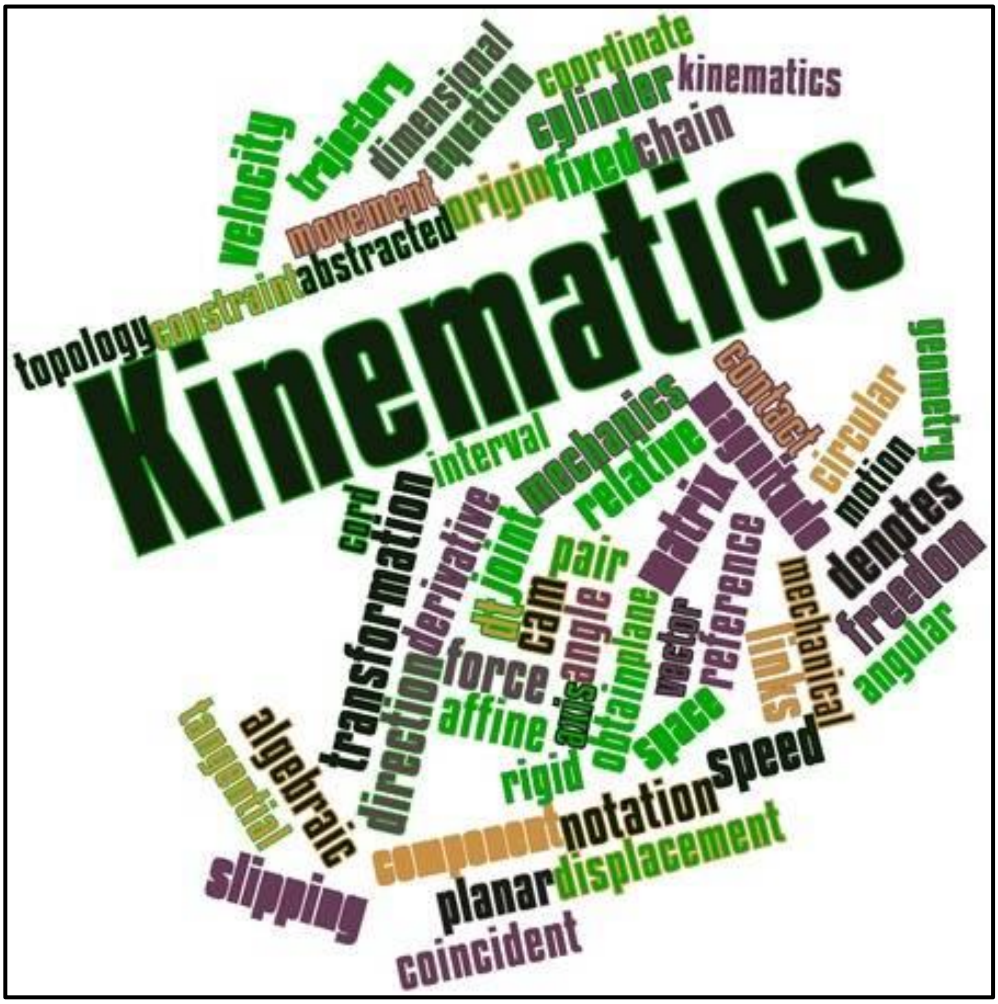
UNIT 3 SPECIALIST MATHS

KINEMATICS



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Chapter 10 Kinematics

Exercise 10A Position, velocity and acceleration

→ **Position** – determined by distance from fixed point 0 (origin)
- direction to right of 0 positive (+ve) · left of 0 negative (−ve)

→ **Displacement** = change in position = $x_{final} - x_{initial} = \Delta x$
(vector quantity) → magnitude + direction

→ **Distance** – measure of length of path taken during change in position
(scalar quantity) → magnitude but no direction

→ Velocity and speed
- **Velocity** (v) → rate of change of position with respect to time (vector)
  Average velocity: $v_{av} = \frac{\text{change in position}}{\text{change in time}} = \frac{\Delta x}{\Delta t} = \frac{x_2 - x_1}{t_2 - t_1}$
  Instantaneous velocity: $v = \frac{dx}{dt} = \dot{x}(t)$
- **Speed** → rate of change of distance with respect to time (scalar)
  → magnitude of velocity
- Average speed = $\frac{\text{distance travelled}}{t_2 - t_1}$
- Instantaneous speed = magnitude of instantaneous velocity
- Convertion of units

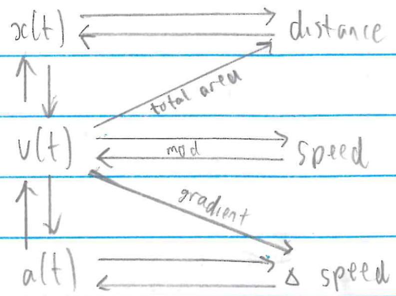
$$m/sec \underset{\times 5/18}{\overset{\times 18/5}{\rightleftarrows}} km/hr$$

→ **Acceleration** (a) → rate of change of velocity with respect to time (vector)
  Average 'a': $a_{av} = \frac{\text{change in velocity}}{\text{change in time}} = \frac{\Delta v}{\Delta t} = \frac{v_2 - v_1}{t_2 - t_1}$
  Instantaneous 'a': $a = \frac{dv}{dt} = \frac{d^2x}{dt^2} = \ddot{x}(t)$
- Acceleration due to gravity ($g = -9.8\ ms^{-2}$ in upwards direction → towards Earth)
- N.B. oscillates – when particle move back + forth btw two points
  – occurs in trig graph (sin + cos)
- during nth second → between $(n+1)$ and 'n' seconds

Exercise 10B: Constant acceleration

→ Formulas for constant acceleration

- $v = u + at$ (No x)
- $x = ut + \frac{1}{2}at^2$ (No v)
- $x = vt - \frac{1}{2}at^2$ (No u)
- $v^2 = u^2 + 2ax$ (No t)
- $x = \frac{1}{2}(u+v)t$ (No a)



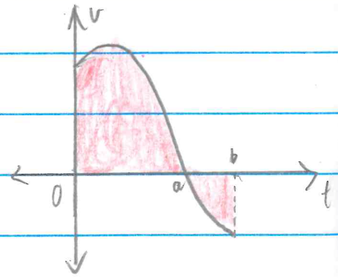
Exercise 10C: Velocity-time graphs

→ Useful information

- Velocity (ordinates): read directly from graph
- Acceleration (gradient)
- Displacement (signed area bounded by graph and t-axis → definite integral)
- Distance (total area 'under' curve) → 'magnitude' of area

- Integral Notation
  - Displacement for $t \in [0,a]$, $= \int_0^a v(t)\,dt$ (+ve)
  - Displacement for $t \in [a,b]$, $= \int_a^b v(t)\,dt$ (−ve)
  - Displacement for $t \in [0,b] = \int_0^b v(t)\,dt$
  - Total distance travelled $= \left|\int_0^a v(t)\,dt\right| + \left|\int_a^b v(t)\,dt\right|$

$$= \int_0^a v(t)\,dt - \int_a^b v(t)\,dt = \int_0^a v(t)\,dt + \int_b^a v(t)\,dt$$



Example 15

A motorist is travelling at a constant speed of 120 km/h when he passes a stationary police car. He continues at that speed for another 15 seconds before uniformly decelerating to 100 km/h in five seconds and then continues with constant velocity. The police car takes off after the motorist the instant it passes. It accelerates uniformly for 25 seconds by which time it has reached 130 km/h. It continues at that speed until it catches up to the motorist. When does the police car catch up to the motorist and how far has it travelled in that time?

NB! Units are MIXED UP!

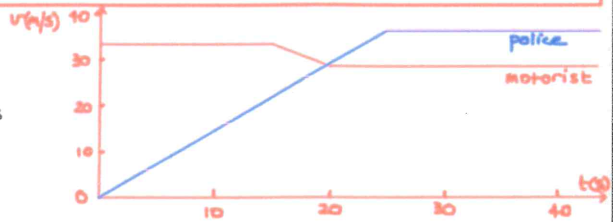
→ Convert km/hr to m/s before starting with velocity-time graph.

Now $x \frac{km}{hr} = x \times \frac{5}{18} \frac{m}{s} = \frac{5x}{18}$ m/s

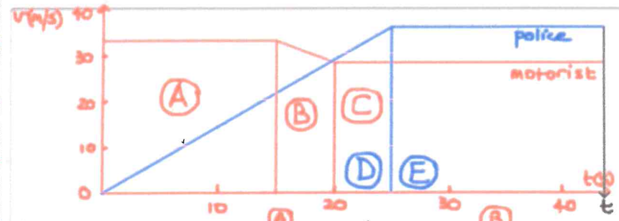
So $120 \frac{km}{hr} = \frac{100}{3}$ m/s. $= 33\frac{1}{3}$

$100 \frac{km}{hr} = \frac{250}{9}$ m/s. $\doteq 27.\dot{7}$

$130 \frac{km}{hr} = \frac{325}{9}$ m/s. $\doteq 36.1$



Let t seconds be the time when both displacements from the time $t=0$ are the same and let x m be that displacement.



Now x (motorist) $= \frac{100}{3} \times 15 + \left(\frac{100}{3} + \frac{250}{9}\right) \times \frac{1}{2} \times 5 + \frac{250}{9} \times (t-20)$

Also, x (police) $= \frac{325}{9} \times 25 \times \frac{1}{2} + \frac{325}{9} \times (t-25)$

Now x(motorist) $=$ x(police) when police catches up

$\rightarrow \frac{100}{\cancel{3}_1} \times \cancel{15}^{5} + \left(\frac{100}{3} + \frac{250}{9}\right) \times \frac{1}{2} \times 5 + \frac{250}{9} \times (t-20) = \frac{325}{9} \times 25 \times \frac{1}{2} + \frac{325}{9} \times (t-25)$

$\rightarrow 500 + \frac{550}{9} \times \frac{5}{2} + \frac{250}{9} \times (t-20) = \frac{8125}{18} + \frac{325}{9}(t-25)$

Mult b.s by 18 :—

$\rightarrow 9000 + 2750 + 500(t-20) = 8125 + 650(t-25)$

$\rightarrow 11750 + 500t - 10000 = 8125 + 650t - 16250$

$\rightarrow 1750 + 500t = 650t - 8125$

$\rightarrow 9875 = 150t \rightarrow t = 395/6 = 65\,5/6$ sec ($65.8\dot{3}$ sec)

Distance travelled when $t = \frac{395}{6}$:—

$x(\text{police}) = \frac{8125}{18} + \frac{325}{9}(t-25)$

Sub $t = \frac{395}{6}$:—

$= \frac{8125}{18} + \frac{325}{9}\left(\frac{395}{6} - 25\right)$

$= \frac{8125}{18} + \frac{325}{9}\left(\frac{395-150}{6}\right)$

$= \frac{8125}{18} + \frac{325}{9}\left(\frac{245}{6}\right)$

$= \frac{52000}{27} \simeq \boxed{1925.93 \text{ m}}$

So car travelled 1.926 km before police caught up.

Exercise 10D Differential equations of form $v = f(x)$ and $a = f(v)$

e.g. 1.

Example 17

The acceleration a, of an object moving along a line is given by $a = -(v+1)^2$ where v is the velocity of the object at time t. Also $v(0) = 10$ and $x(0) = 0$ (x is the position of the particle at time t). Find:

a an expression for the velocity of the object in terms of t

b an expression for the position of the object in terms of t

(a) $a = \frac{dv}{dt} = -(v+1)^2$

$\rightarrow \frac{dt}{dv} = -\frac{1}{(v+1)^2}$

$\rightarrow t = -\int \frac{1}{(v+1)^2}\,dv$

$= +\frac{1}{(v+1)} + c$

Sub $v = 10$, $t = 0$

$\rightarrow 0 = \frac{1}{11} + c$

$\rightarrow c = -\frac{1}{11}$

So $t = \frac{1}{v+1} - \frac{1}{11}$

$\rightarrow 11t(v+1) = 11 - (v+1)$

$\rightarrow 11tv + 11t = 10 - v$

$\rightarrow v(11t+1) = 10 - 11t$

$\rightarrow \boxed{v = \frac{10-11t}{11t+1}}$

(b) $v = \frac{dx}{dt} = \frac{10-11t}{1+11t}$

$\rightarrow x = \int \frac{-11t+10}{11t+1}\,dt$

$= -\int \left(\frac{11t+1-11}{11t+1}\right) dt$

i.e. $x = -\int \left(1 - \frac{11}{11t+1}\right) dt$

$= -t + \log_e|11t+1| + c$

Sub $t = 0$, $x = 0$:—

$\rightarrow 0 = 0 + \cancel{\log_e(1)}^{=0} + c$

$\rightarrow c = 0$

Hence, $\boxed{x = \log_e(11t+1) - t, \; t \geq 0}$

(a) $v = \frac{11}{11t+1} - 1$

e.g. 2.

Example 18

A body moves in a straight line with an initial velocity of 25 m/s so that its acceleration a m/s^2 is given by $a = -k(50 - v)$, where k is a positive constant, and v m/s is its velocity at a given instant. Find v in terms of t and sketch the velocity–time graph for the motion. (The motion stops when the body is instantaneously at rest for the first time.)

$a = -k(50-v)$

We need $v(t)$:—

So $\frac{dv}{dt} = -k(50-v)$

$\rightarrow \frac{dt}{dv} = \frac{-1}{k(50-v)}$

$= \frac{1}{k(v-50)}$

$\rightarrow t = \frac{1}{k}\log_e|v-50| + c$

Sub $t = 0$, $v = 25$:—

$\rightarrow 0 = \frac{1}{k}\log_e(25) + c$

$\rightarrow c = -\frac{1}{k}\log_e(25)$

Hence, $t = \frac{1}{k}\log_e\left|\frac{v-50}{25}\right|$

So $\frac{v-50}{25} = Ae^{kt}$ where A is a constant

Sub $v = 25$, $t = 0$:—

$\rightarrow -1 = A$

Hence, $\frac{v-50}{25} = -e^{kt}$

$\rightarrow \boxed{v = -25e^{kt} + 50}$

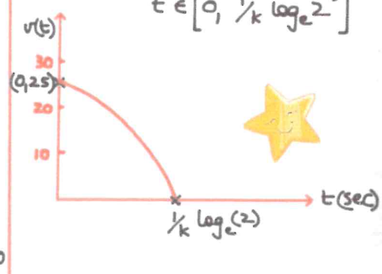
If k is a +ve number, then $e^{kt} \uparrow$ as $t \uparrow$

So, when $v = 0$, $0 = -25e^{kt} + 50$

$\rightarrow e^{kt} = 2 \rightarrow kt = \log_e 2$

$\rightarrow \boxed{t = 1/k \log_e 2}$

So $v = -25e^{kt} + 50$, $k > 0$, $t \in [0, 1/k \log_e 2]$



- e.g. 3. The brakes are applied in a car travelling in a straight line. The acceleration, a m/s^2, of the car is given by $a = -0.4\sqrt{225 - v^2}$. If the initial velocity of the car was 12 m/s, find an expression for v in terms of t, the time after the brakes were first applied.

$\rightarrow a = \frac{dv}{dt} = -\frac{2}{5}\sqrt{225 - v^2} \rightarrow \frac{dt}{dv} = \frac{-5}{2\sqrt{225 - v^2}}$

$t = \frac{5}{2}\int \frac{-1}{\sqrt{225 - v^2}}\, dv = \frac{5}{2}\arccos\left(\frac{v}{15}\right) + c$

Sub $t = 0$, $v = 12$: $0 = \frac{5}{2}\arccos\left(\frac{4}{5}\right) + c \rightarrow c = -\frac{5}{2}\arccos\left(\frac{4}{5}\right)$

$t = \frac{5}{2}\arccos\left(\frac{v}{15}\right) - \frac{5}{2}\arccos\left(\frac{4}{5}\right)$

$\frac{5}{2}\arccos\left(\frac{v}{15}\right) = t + \frac{5}{2}\arccos\left(\frac{4}{5}\right)$

$\therefore \boxed{v = 15\cos\left(\frac{2}{5}t + \arccos\left(\frac{4}{5}\right)\right)}$

Exercise 10E Other expressions for acceleration

$\rightarrow a = f(v) \rightarrow a = v\left(\frac{dv}{dx}\right)$ Calc: $\int_{v_1}^{v_2} v\,dv = \int_{x_1}^{x_2} \frac{dv}{dx}\,dx$

$\rightarrow a = f(x) \rightarrow a = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

- Complex example 1.

Example 22

An object falls from a hovering helicopter over the ocean 1000 m above sea level. Find the velocity of the object when it hits the water:

a neglecting air resistance b assuming air resistance is $0.2v^2$.

(a) ↑ +ve direction

Now $a = -9.8$ m/sec^2

$u = 0$

$v = ?$

$x = -1000$

So $v^2 = u^2 + 2ax$

$\rightarrow v^2 = 0^2 - 2 \times 9.8 \times (-1000)$

$\rightarrow v^2 = 19600$

$\rightarrow \boxed{v = -140 \text{ m/s}}$

(b) Assume it means acceleration from wind resistance is $+\frac{1}{5}v^2$

So $a = -9.8 + \frac{1}{5}v^2$

$v = 0$ when $x = 0$

So $v\frac{dv}{dx} = -9.8 + \frac{v^2}{5}$

$= \frac{v^2 - 49}{5}$

$\rightarrow \frac{dv}{dx} = \frac{v^2 - 49}{5v}$

$\rightarrow \frac{dx}{dv} = \frac{5v}{v^2 - 49}$

$\rightarrow x = \int \frac{5v\,dv}{v^2 - 49}$

$= \frac{5}{2}\int \frac{2v\,dv}{v^2 - 49}$

$= \frac{5}{2}\log_e|v^2 - 49| + c$

Sub $x = 0$, $v = 0$

$\rightarrow 0 = \frac{5}{2}\log_e(49) + c$

$\rightarrow c = -\frac{5}{2}\log_e(49)$

Hence, $x = \frac{5}{2}\log_e\left|\frac{v^2 - 49}{49}\right|$

So $\frac{2x}{5} = \log_e\left|\frac{v^2 - 49}{49}\right|$

$\rightarrow \frac{v^2 - 49}{49} = Ae^{2x/5}$ where A is a constant

Sub $v = 0$ when $x = 0$:—

$\rightarrow -1 = A$

$\rightarrow \frac{v^2 - 49}{49} = -e^{2x/5}$

$\rightarrow v^2 = 49(1 - e^{2x/5})$

$\rightarrow v = \pm 7\sqrt{1 - e^{2x/5}}$

$v = \pm 7\sqrt{1 - e^{2x/5}}$

Now v must always be negative because up was taken to be +ve at the start and the object is moving downwards.

$v = -7\sqrt{1 - e^{2x/5}}$

Finally, when $x = -1000$,

$\boxed{v = -7\sqrt{1 - e^{-400}} \text{ m/s}}$

(As e^{-400} is SO SMALL, $v \approx -7$ m/sec).

- e.g. 2. A missile is fired vertically upwards from a point on the ground level with the base of a tower 64 m high. The missile is level with the top of the tower 0.8 seconds after being fired. Let g m/s^2 be acceleration due to gravity. Find in terms of g:

(a) the initial velocity of missile

(b) the time taken to reach its greatest height

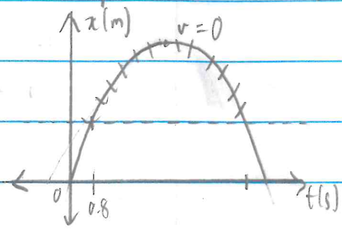
(c) the greatest height

(d) the length of time for which the missile is higher than top of tower

(a) $x = ut + \frac{1}{2}at^2 \to h = ut - \frac{1}{2}gt^2$ ... ① (since $g = -9.8\ ms^{-2}$ in upwards direction)

$\to u = \frac{h}{t} + \frac{1}{2}gt = \frac{64}{0.8} + \frac{1}{2}g(0.8)$

$= \boxed{(80 + \frac{2}{5}g)\ m/s}$



(b) $v = u + at \to u - gt = 0 \to t = \frac{u}{g}$

$\to t = \boxed{(\frac{80}{g} + \frac{2}{5})\ s}$ ... ②

(c) Sub ② into ①, $h = (80 + \frac{2}{5}g)(\frac{80}{g} + \frac{2}{5}) - \frac{1}{2}g(\frac{80}{g} + \frac{2}{5})^2$

$= \frac{1}{g}(80 + \frac{2}{5}g)^2 - \frac{1}{2g}(80 + \frac{2}{5}g)^2 = \boxed{\frac{1}{2g}(80 + \frac{2}{5}g)^2\ m}$

(d) Total time $= 2(\frac{80}{g} + \frac{2}{5} - \frac{4}{5})\ s$

$= 2(\frac{80}{g} - \frac{2}{5})\ s = \boxed{(\frac{160}{g} - \frac{4}{5})\ s}$

- e.g. 3. VCAA 2017 Exam 2 Q2

A helicopter is hovering at a constant height above a fixed location. A skydiver falls from rest for two seconds from the helicopter. The skydiver is subject only to gravitational acceleration and air resistance is negligible for the first two seconds. Let downward displacement be positive.

(a) Find the distance, in metres, fallen in first two seconds.

$x = ut + \frac{1}{2}at^2 = \frac{1}{2}gt^2$

$= \frac{1}{2}(9.8)(2)^2 = \boxed{19.6\ m}$

(b) Show that the speed of the skydiver after two seconds is $19.6\ ms^{-1}$.

$a = \frac{\Delta v}{\Delta t} \to v = at \to v = gt = 9.8 \times 2 = \boxed{19.6\ m/s}$ (Shown)

After two seconds, air resistance is significant and the acceleration of the skydiver is given by $a = g - 0.01v^2$.

(c) Find the limiting (terminal) velocity, in ms^{-1}, that the skydiver reaches.

$\to a = 9.8 - 0.01v^2 = 0$

$v^2 = 980 \to v = \sqrt{980} = \boxed{14\sqrt{5}\ m/s}$ (or $10\sqrt{g}\ m/s$)

(d)(i) Write down an expression involving a definite integral that gives the time taken for the skydiver to reach a speed of $30\ ms^{-1}$.

$\to \frac{dv}{dt} = 9.8 - 0.01v^2 \to \frac{dt}{dv} = \frac{1}{9.8 - 0.01v^2} \to t = \int_{19.6}^{30} \frac{1}{9.8 - 0.01v^2}\,dv$

$\therefore T = \boxed{2 + \int_{19.6}^{30} \frac{1}{9.8 - 0.01v^2}\,dv}$

(ii) Hence, evaluate this integral, correct to nearest tenth of a second.

$\to t = \boxed{5.8\ s}$

(e) Write down an expression involving a definite integral that gives the distance through which the skydiver falls to reach a speed of 30 ms^{-1}. Find this distance, correct to the nearest metre.

$\rightarrow v\left(\frac{dv}{dx}\right) = 9.8 - 0.01v^2 \rightarrow \frac{dv}{dx} = \frac{9.8-0.01v^2}{v} = \frac{980-v^2}{100v} \rightarrow \frac{dx}{dv} = \frac{100v}{980-v^2}$

$\downarrow$

$x = \int_{19.6}^{30} \frac{100}{980-v^2}\, dv$

Distance $= \boxed{19.6 + \int_{19.6}^{30}\left(\frac{100v}{980-v^2}\right) dv}$

$\rightarrow$ Distance $= 119.99727\ldots \rightarrow \boxed{120 \text{ metres}}$

Miscellenous Examples

A lift is travelling downwards at a constant speed of v ms^{-1}. A woman standing in the lift drops her cup of coffee from a height of h metres above the floor of the lift. The coffee cup hits the floor of the lift after 0.5 seconds. (Ex 10B)

Find the value of h.

Lift floor: $u = v$ m/s, $a = 0$ m/s^2, $t = \frac{1}{2}$ s $x_L(t)$ = downwards displacement of lift floor

$x = ut + \frac{1}{2}at^2 = \frac{1}{2}v$

Coffee cup: $u = v$ m/s, $a = g$ m/s^2, $t = \frac{1}{2}$ s $x_c(t)$ = downwards displacement of cup

$x = ut + \frac{1}{2}at^2 = \frac{1}{2}v + \frac{g}{8}$

$h = x_c\left(\frac{1}{2}\right) - x_L\left(\frac{1}{2}\right) = \boxed{\frac{g}{8} \text{ metres}}$

The acceleration of a particle at time t seconds is given by $\ddot{\underset{\sim}{r}}(t) = 2\underset{\sim}{i} + 4\underset{\sim}{j}$ ms^{-2}.

Find the distance of the particle from its initial position when $t = 2$, given that $\dot{\underset{\sim}{r}}(0) = -2\underset{\sim}{i} - 3\underset{\sim}{k}$ ms^{-1} and $\underset{\sim}{r}(0) = \underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}$ m. (Ex 12D)

$\dot{\underset{\sim}{r}}(t) = \int \ddot{\underset{\sim}{r}}(t)\, dt = 2t\underset{\sim}{i} + 4t\underset{\sim}{j} + \underset{\sim}{c}$

Sub $\dot{\underset{\sim}{r}}(0) = -2\underset{\sim}{i} - 3\underset{\sim}{k}$, $\underset{\sim}{c} = -2\underset{\sim}{i} - 3\underset{\sim}{k}$

$\dot{\underset{\sim}{r}}(t) = (2t-2)\underset{\sim}{i} + 4t\underset{\sim}{j} - 3\underset{\sim}{k}$

$\underset{\sim}{r}(t) = \int \dot{\underset{\sim}{r}}(t)\, dt = (t^2 - 2t)\underset{\sim}{i} + 2t^2\underset{\sim}{j} - 3t\underset{\sim}{k} + \underset{\sim}{d}$

Sub $\underset{\sim}{r}(0) = \underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}$, $\underset{\sim}{d} = \underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}$

$\underset{\sim}{r}(t) = (t^2 - 2t + 1)\underset{\sim}{i} + (2t^2 - 1)\underset{\sim}{j} + (2 - 3t)\underset{\sim}{k}$

$\underset{\sim}{r}(2) = (4 - 4 + 1)\underset{\sim}{i} + (2(4) - 1)\underset{\sim}{j} + (2 - 6)\underset{\sim}{k}$

$= \underset{\sim}{i} + 7\underset{\sim}{j} - 4\underset{\sim}{k}$

Distance $= |\underset{\sim}{r}(2) - \underset{\sim}{r}(0)| = \sqrt{(1-1)^2 + (7-(-1))^2 + (-4-2)^2}$

$= \sqrt{8^2 + (-6)^2} = \boxed{10 \text{ metres}}$

(N.B. not $|\underset{\sim}{r}(2)| = \sqrt{1^2 + 7^2 + (-4)^2}$)