3.4 Differentiate the following functions with respect to x.

(a) $y = \dfrac{1+x}{2+x}$ (b) $y = \dfrac{x^2+1}{x}$ (c) $y = \dfrac{\ln x}{\cos x}$

(d) $y = \dfrac{e^z - e^{-z}}{e^z + e^{-z}}$ (e) $y = \dfrac{x^{\frac{1}{3}} + x^{-\frac{1}{3}}}{\tan 2x}$ (f) $y = \dfrac{1+\sin x}{1+\cos x}$

Outline Solution

(a) Using the quotient rule for $y = \dfrac{1+x}{2+x}$:

$$\begin{aligned} u &= 1+x \quad \Rightarrow \quad \frac{du}{dx} = 1 \qquad\qquad v = 2+x \quad \Rightarrow \quad \frac{dv}{dx} = 1 \\ \frac{dy}{dx} &= \frac{v\,\frac{du}{dx} - u\,\frac{dv}{dx}}{v^2} = \frac{(2+x)\cdot 1 - (1+x)\cdot 1}{(2+x)^2} \\ &= \frac{1}{(2+x)^2} \end{aligned}$$

(b) Using the quotient rule for $y = \dfrac{x^2+1}{x}$:

$$\begin{aligned} u &= x^2+1 \quad \Rightarrow \quad \frac{du}{dx} = 2x \qquad\qquad v = x \quad \Rightarrow \quad \frac{dv}{dx} = 1 \\ \frac{dy}{dx} &= \frac{v\,\frac{du}{dx} - u\,\frac{dv}{dx}}{v^2} = \frac{x\cdot 2x - \left(x^2+1\right)\cdot 1}{x^2} \\ &= \frac{x^2-1}{x^2} \end{aligned}$$

(c) Using the quotient rule for $y = \dfrac{\ln x}{\cos x}$:

$$\begin{aligned} u &= \ln x \quad \Rightarrow \quad \frac{du}{dx} = \frac{1}{x} \qquad\qquad v = \cos x \quad \Rightarrow \quad \frac{dv}{dx} = -\sin x \\ \frac{dy}{dx} &= \frac{v\,\frac{du}{dx} - u\,\frac{dv}{dx}}{v^2} = \frac{\cos x\left(\frac{1}{x}\right) - \ln x(-\sin x)}{\cos^2 x} \\ &= \frac{\cos x + x\ln x\sin x}{x\cos^2 x} \end{aligned}$$

(d) Using the quotient rule for $y = \dfrac{e^z - e^{-z}}{e^z + e^{-z}}$:

$$\begin{aligned} u &= e^z - e^{-z} \quad \Rightarrow \quad \frac{du}{dx} = e^z + e^{-z} \qquad\qquad v = e^z + e^{-z} \quad \Rightarrow \quad \frac{dv}{dx} = e^z - e^{-z} \\ \frac{dy}{dx} &= \frac{v\,\frac{du}{dx} - u\,\frac{dv}{dx}}{v^2} = \frac{(e^z+e^{-z})\cdot(e^z+e^{-z}) - (e^z-e^{-z})\cdot(e^z-e^{-z})}{\left(e^z+e^{-z}\right)^2} \\ &= \frac{\left(e^{2z}+2+e^{-2z}\right) - \left(e^{2z}-2+e^{-2z}\right)}{\left(e^z+e^{-z}\right)^2} \\ &= \frac{4}{\left(e^z+e^{-z}\right)^2} \end{aligned}$$

(e) Using the quotient rule for $y = \dfrac{x^{\frac{1}{3}} + x^{-\frac{1}{3}}}{\tan 2x}$:

$$u = x^{\frac{1}{3}} + x^{-\frac{1}{3}} \Rightarrow \frac{du}{dx} = \frac{1}{3}x^{-\frac{2}{3}} - \frac{1}{3}x^{-\frac{4}{3}} \qquad v = \tan(2x) \Rightarrow \frac{dv}{dx} = 2\sec^2(2x)$$

$$\frac{dy}{dx} = \frac{v\,\frac{du}{dx} - u\,\frac{dv}{dx}}{v^2} = \frac{\tan 2x\left[\frac{1}{3}x^{-\frac{2}{3}} - \frac{1}{3}x^{-\frac{4}{3}}\right] - \left(x^{\frac{1}{3}} + x^{-\frac{1}{3}}\right)2\sec^2 2x}{\tan^2 2x}$$

$$= \frac{\tan 2x\left(x^{-\frac{2}{3}} - x^{-\frac{4}{3}}\right) - 6\left(x^{\frac{1}{3}} + x^{-\frac{1}{3}}\right)\sec^2 2x}{3\tan^2 2x}$$

(f) Using the quotient rule for $y = \dfrac{1+\sin x}{1+\cos x}$:

$$u = 1 + \sin x \Rightarrow \frac{du}{dx} = \cos x \qquad v = 1 + \cos x \Rightarrow \frac{dv}{dx} = -\sin x$$

$$\frac{dy}{dx} = \frac{v\,\frac{du}{dx} - u\,\frac{dv}{dx}}{v^2} = \frac{(1+\cos x)(\cos x) - (1+\sin x)(-\sin x)}{(1+\cos x)^2}$$

$$= \frac{\cos x + \cos^2 x + \sin x + \sin^2 x}{(1+\cos x)^2}$$

$$= \frac{\cos x + \sin x + 1}{(1+\cos x)^2}$$