MATRICES & TRANSFORMATIONS

WORKSHEET 2 – IMAGES OF EQUATIONS

QUESTION 1

Find the equation of the image of the line $x+y=1$ under the transformation defined by the matrix $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Solution

QUESTION 2

The line $y=2x-1$ undergoes a translation with matrix $\begin{bmatrix} -3 \\ -2 \end{bmatrix}$. Find the equation of the image.

Solution

QUESTION 3

The parabola $y = x^2 + 1$ is transformed by the matrix $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$. Find the equation of the image.

Solution

QUESTION 4

Consider the linear transformation represented by the matrix $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$. Find the image of the curve $x^2 - y^2 = 1$ under this transformation.

Solution

QUESTION 5

The linear transformation T is defined by the equation $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

Find the equation of the image of the curve $x^2 + y^2 = 4$ under T.

Solution

QUESTION 6

The linear transformation T is defined by the equation $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 4 \end{bmatrix}$.

Find the equation of the image of the curve $y = \sin x$ under T.

Solution

QUESTION 7

Under the linear transformation of the plane $T : R^2 \to R^2$ is defined by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

Find the equation of the image of $y = \log_e x$ as the result of this linear transformation.

Solution

QUESTION 8

The linear transformation T is defined by the equation $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ -2 \end{bmatrix}$.

Find the equation of the image of the curve $y = e^x$ under T.

Solution

QUESTION 9

The function $y = \cos x$ is transformed to produce the graph of $y = -3\cos(2x+1)$.

Write a matrix equation, $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$, to describe the linear transformation that occurred. Hence find the equation of the image of the curve $y = e^x$ under T.

Solution

QUESTION 10

The function $y = e^x$ is transformed to produce the graph of $y = \frac{1}{2}e^{1-4x} + 5$.

Write a matrix equation, $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$, to describe the linear transformation that occurred. Hence find the equation of the image of the curve $y = (x-1)^2$ under T.

Solution

SOLUTIONS

QUESTION 1

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x+1 \\ y+1 \end{bmatrix}$$

$\therefore x' = x + 1 \qquad y' = y + 1$

$\therefore x = x' - 1 \qquad \therefore y = y' - 1$

$$x + y = 1$$

$$\therefore (x'-1)+(y'-1) = 1$$

$$x' + y' - 2 = 1$$

$$x' + y' = 3$$

$$\therefore x + y = 3$$

QUESTION 2

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -3 \\ -2 \end{bmatrix} = \begin{bmatrix} x-3 \\ y-2 \end{bmatrix}$$

$\therefore x' = x - 3 \qquad y' = y - 2$

$x = x' + 3 \qquad y = y' + 2$

$$\therefore y = 2x - 1$$

$$(y'+2) = 2(x'+3) - 1$$

$$y' + 2 = 2x' + 6 - 1$$

$$y' = 2x' + 3$$

$$y = 2x + 3$$

QUESTION 3

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ y \end{bmatrix}$$

$\therefore x' = 2x \qquad y' = y$

$\therefore x = \frac{x'}{2} \qquad \therefore y = y'$

$y = x^2 + 1$

$\therefore y' = \left(\frac{x'}{2}\right)^2 + 1$

$y' = \left(\frac{x'^2}{4}\right) + 1$

$\therefore y = \frac{x^2}{4} + 1$

QUESTION 4

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ 3y \end{bmatrix}$$

$\therefore x' = 2x \qquad y' = 3y$

$x = \frac{x'}{2} \qquad y = \frac{y'}{3}$

$\therefore x^2 - y^2 = 1$

$\left(\frac{x}{2}\right)^2 - \left(\frac{y}{3}\right)^2 = 1$

$\frac{x^2}{4} - \frac{y^2}{9} = 1$

QUESTION 5

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ 2y \end{bmatrix} + \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} x+3 \\ 2y+2 \end{bmatrix}$$

$\therefore \ x' = x + 3 \qquad y' = 2y + 2$

$\therefore \ x = x' - 3 \qquad \therefore \ y = \dfrac{y'-2}{2}$

$$x^2 + y^2 = 4$$

$$(x'-3)^2 + \left(\frac{y'-2}{2}\right)^2 = 4$$

$$(x-3)^2 + \frac{(y-2)^2}{4} = 4$$

QUESTION 6

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 2x \\ 3y \end{bmatrix} + \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 2x+1 \\ 3y+4 \end{bmatrix}$$

$\therefore \ x' = 2x + 1 \qquad y' = 3y + 4$

$x = \dfrac{x'-1}{2} \qquad y = \dfrac{y'-4}{3}$

$$y = \sin x$$

$$\frac{y-4}{3} = \sin\left(\frac{x-1}{2}\right)$$

$$\therefore \ y = 3\sin\left(\frac{x-1}{2}\right) + 4$$

QUESTION 7

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \frac{1}{3}x \\ -\frac{1}{2}y \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$\therefore x' = \frac{x}{3}$

$\therefore x = 3x'$

$\therefore y' = -\frac{1}{2}y + 2$

$\therefore y' - 2 = -\frac{1}{2}y$

$\therefore 2y' - 4 = -y$

$\therefore y = 4 - 2y'$

Substitute into $y = \log_e x$:

$$4 - 2y' = \log_e(3x')$$

$$-2y' = \log_e(3x') - 4$$

$$2y' = 4 - \log_e(3x')$$

$$y' = 2 - \frac{1}{2}\log_e(3x')$$

$$\therefore y = 2 - \frac{\log_e(3x)}{2}$$

QUESTION 8

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} 2x \\ -3y \end{bmatrix} + \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} 2x \\ -3y-2 \end{bmatrix}$$

$\therefore\ x' = 2x$ $\qquad\qquad \therefore\ y' = -3y-2$

$\therefore\ x = \dfrac{x'}{2}$ $\qquad\qquad -3y = y'+2$

$$y = \frac{-y'-2}{3}$$

Substitute into $y = e^{x}$:

$$\frac{-y'-2}{3} = e^{x'/2}$$

$$-y'-2 = 3e^{x'/2}$$

$$-y' = 3e^{x'/2} + 2$$

$$y = -3e^{x/2} - 2$$

QUESTION 9

Dilation of factor 3 from x axis : $\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$

Dilation of factor $\frac{1}{2}$ from y axis : $\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}$

Reflection in x axis : $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

Translation of $\frac{1}{2}$ units to the left : $\begin{bmatrix} -\frac{1}{2} \\ 0 \end{bmatrix}$

$$T = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -1/2 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1/2\, x \\ -3y \end{bmatrix} + \begin{bmatrix} -1/2 \\ 0 \end{bmatrix}$$

$\therefore x' = \frac{x}{2} - \frac{1}{2}$

$2x' = x - 1$

$\therefore x = 2x' + 1$

$\therefore y' = -3y$

$\therefore y = -\frac{y'}{3}$

substitute into $y = e^{x}$:

$-\frac{y'}{3} = e^{2x'+1}$

$-y' = 3e^{2x'+1}$

$y = -3e^{2x+1}$

QUESTION 10

Dilation of $\frac{1}{2}$ from x axis: $\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}$

Dilation of factor $\frac{1}{4}$ from y axis: $\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & 1 \end{bmatrix}$

Reflection in Y axis: $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

Translation of $\frac{1}{4}$ units to right: $\begin{bmatrix} 1/4 \\ 0 \end{bmatrix}$

Translation of 5 units up: $\begin{bmatrix} 0 \\ 5 \end{bmatrix}$

$$T = \begin{bmatrix} -\frac{1}{4} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \frac{1}{4} \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -\frac{x}{4} \\ \frac{1}{2}y \end{bmatrix} + \begin{bmatrix} \frac{1}{4} \\ 5 \end{bmatrix}$$

$x' = -\frac{x}{4} + \frac{1}{4}$ $\qquad$ $y' = \frac{1}{2}y + 5$

$4x' = -x + 1$ $\qquad$ $y' = \frac{y+10}{2}$

$4x' - 1 = -x$ $\qquad$ $2y' = y + 10$

$\therefore x = 1 - 4x'$ $\qquad$ $y = 2y' - 10$

substitute into $y = (x-1)^2$

$2y' - 10 = (1 - 4x' - 1)^2$

$2y' - 10 = (-4x')^2$

$2y' = 16(x')^2 + 10$

$\therefore y' = 8(x')^2 + 5$ $\qquad$ $\therefore y = 8x^2 + 5$