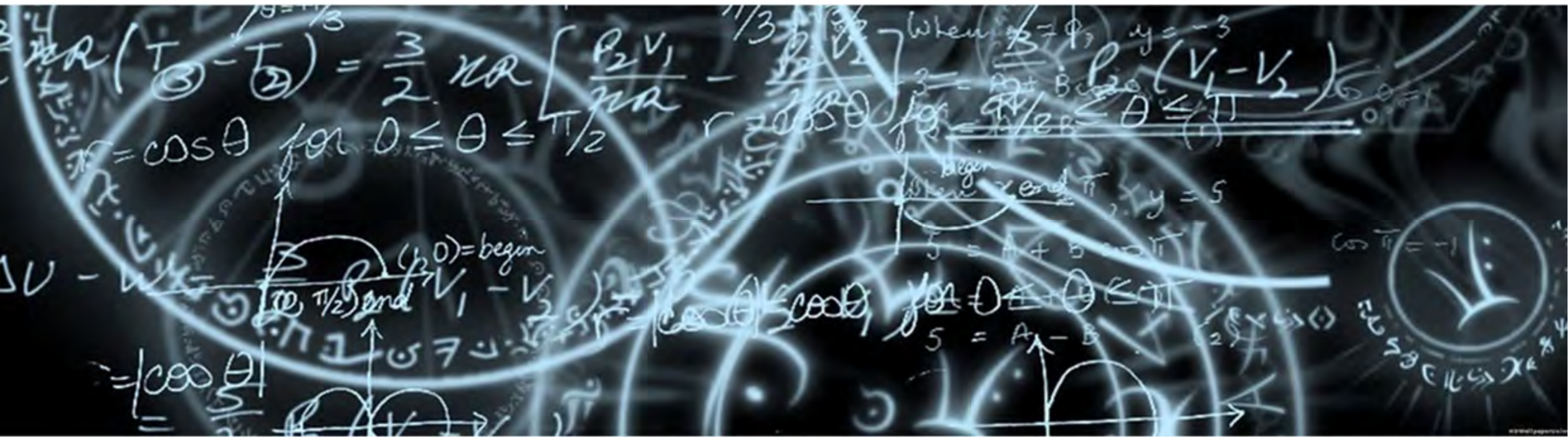
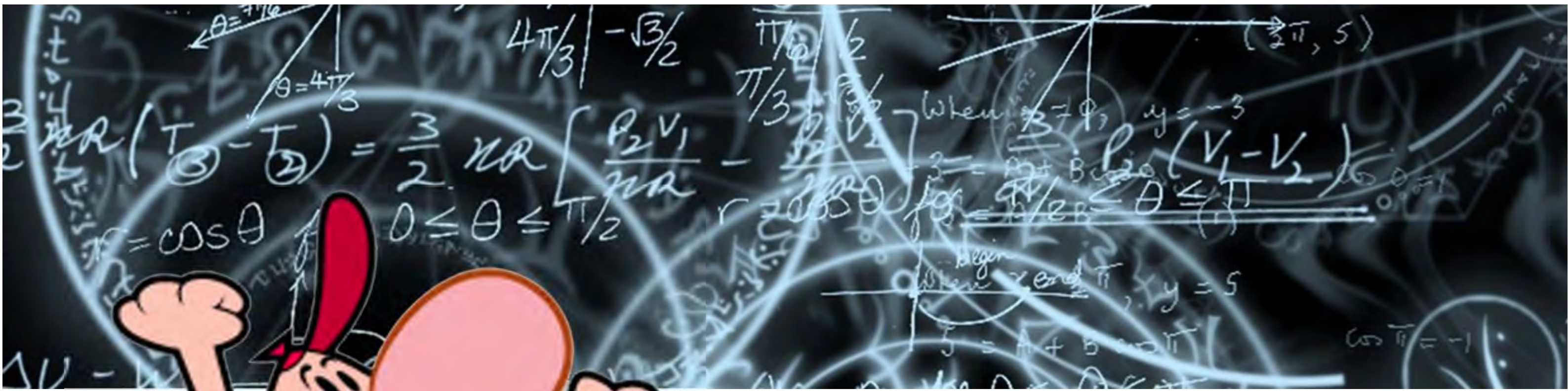


UNIT 3 SPECIALIST MATHS

CIRCULAR FUNCTIONS

REVISION NOTES FOR YOUR SACS & EXAMS



WRITTEN BY A STUDENT WHO OBTAINED A SCALED STUDY SCORE OF 52.46!

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VCAA 2014 Exam 2 Q2

Question 2 (13 marks)

Consider the complex number $z_1 = \sqrt{3} - 3i$.

a. i. Express z_1 in polar form. 2 marks

$|z_1| = \sqrt{(\sqrt{3})^2 + (-3)^2} = 2\sqrt{3}$

$\tan(\theta) = \frac{3}{\sqrt{3}} = \sqrt{3} \to \theta = \frac{\pi}{3} \to Arg(z_1) = -\frac{\pi}{3}$

$\sqrt{3}$, θ, 3

$\therefore \boxed{z_1 = 2\sqrt{3}\,\text{cis}\left(-\frac{\pi}{3}\right)}$

ii. Find $\text{Arg}(z_1^{\,4})$. 1 mark

$Arg(z_1^{\,4}) = -\frac{4\pi}{3} \to \boxed{\frac{2\pi}{3}}$

iii. Given that $z_1 = \sqrt{3} - 3i$ is one root of the equation $z^3 + 24\sqrt{3} = 0$, find the other two roots, expressing your answers in cartesian form. 2 marks

$P(z) = z^3 + 24\sqrt{3} = (z - (\sqrt{3} - 3i))(z - (\sqrt{3} + 3i))(z - \alpha)$

$= ((z - \sqrt{3})^2 + 9)(z - \alpha) = (z^2 - 2\sqrt{3}z + 12)(z - \alpha)$

Equating coefficients $24\sqrt{3} = -12\alpha \to \alpha = -2\sqrt{3}$

Other two roots, $\boxed{z = \sqrt{3} + 3i, \ z = -2\sqrt{3}}$

b. i. Find the value of $(z_1 + 2i)(\overline{z}_1 - 2i)$, where $z_1 = \sqrt{3} - 3i$. 1 mark

4

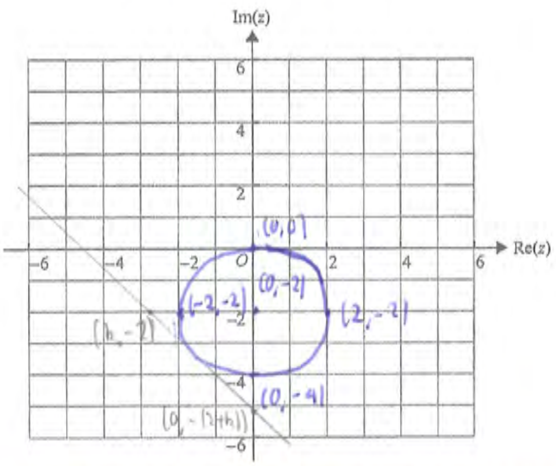
ii. Show that the relation $(z + 2i)(\overline{z} - 2i) = 4$ can be expressed in cartesian form as $x^2 + (y + 2)^2 = 4$. 2 marks

$(x + yi + 2i)(x - yi - 2i) = 4 \to (x + (y+2)i)(x - (y+2)i) = 4$

$x^2 - (y+2)^2 i^2 = 4$

$\boxed{x^2 + (y+2)^2 = 4}$

iii. Sketch $\{z : (z + 2i)(\overline{z} - 2i) = 4\}$ on the axes below. 2 marks



c. The line joining the points corresponding to $k - 2i$ and $-(2 + k)i$, where $k < 0$, is tangent to the curve given by $\{z : (z + 2i)(\overline{z} - 2i) = 4\}$.

Find the value of k. 3 marks

Strategy :—

① Find ~~equation~~ equation of line joining $k - 2i$ and $-(2+k)i$

② Set up an equation for the intersection of the line and the circle.

③ Solve the equation above for only one solution point (tangent)

So the two points are $(k, -2)$ and $(0, -(2+k))$

$m_{line} = \dfrac{-2 - (-(2+k))}{k - 0}$

$= \dfrac{-k}{k} = -1$

$\to$ Eq^{n} of line passing through $(k, -2)$ is

$y - y_1 = m(x - x_1)$

ie $y - (-2) = -1(x - k)$

$\to y = -x + (k - 2)$ — ①

Also, $x^2 + (y+2)^2 = 4$ — ②

Sub for y from ① into ② :—

② $\to x^2 + (-x + k \cancel{-2} \cancel{+2})^2 = 4$

$\to x^2 + (x - k)^2 = 4$

$\to x^2 + x^2 - 2kx + k^2 = 4$

$\to 2x^2 - (2k)x + (k^2 - 4) = 0$

$\Delta = (-2k)^2 - 4 \times 2 \times (k^2 - 4) = 0$

$\to 4k^2 - 8(k^2 - 4) = 0$

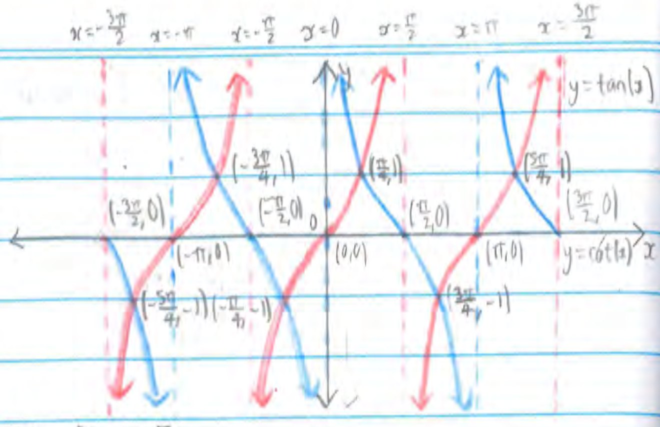
$\to -4k^2 + 32 = 0$

$\to k^2 = 8 \to \boxed{k = -2\sqrt{2}} \ (k < 0)$

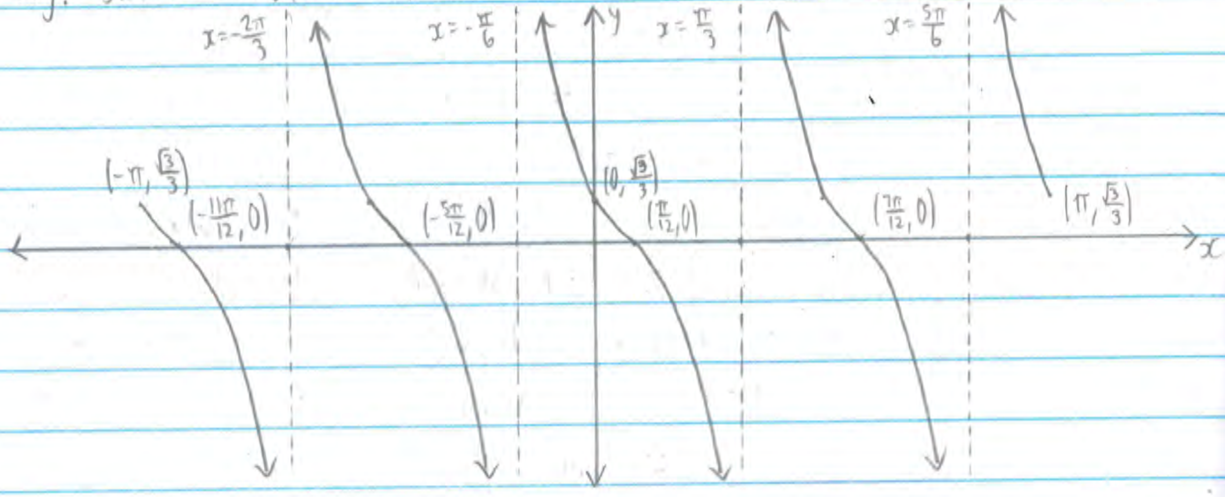
→ Cotangent function: $y = \cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}$

$= \frac{1}{\tan(\theta)} \quad (\tan(\theta) \neq 0)$

$= -\tan(\theta + \frac{\pi}{2})$



- Domain: $\mathbb{R} \setminus \{n\pi, n \in \mathbb{Z}\}$
- Range: $\mathbb{R}$
- Asymptotes: $\theta = n\pi, n \in \mathbb{Z}$
- e.g. Sketch $\cot(2x - \frac{2\pi}{3})$ over interval $[-\pi, \pi]$



→ Useful symmetry properties

Quadrant 2	Quadrant 3	Quadrant 4
$\operatorname{cosec}(\pi - x) = \operatorname{cosec}(x)$	$\operatorname{cosec}(\pi + x) = -\operatorname{cosec}(x)$	$\operatorname{cosec}(2\pi - x) = -\operatorname{cosec}(x)$
$\sec(\pi - x) = -\sec(x)$	$\sec(\pi + x) = -\sec(x)$	$\sec(2\pi - x) = \sec(x)$
$\cot(\pi - x) = -\cot(x)$	$\cot(\pi + x) = \cot(x)$	$\cot(2\pi - x) = -\cot(x)$

→ Complementary properties

Quad 1	Quad 2	Quad 3	Quad 4
$\operatorname{cosec}(\frac{\pi}{2} - x) = \sec(x)$	$\operatorname{cosec}(\frac{\pi}{2} + x) = \sec(x)$	$\operatorname{cosec}(\frac{3\pi}{2} - x) = -\sec(x)$	$\operatorname{cosec}(\frac{3\pi}{2} + x) = -\sec(x)$
$\sec(\frac{\pi}{2} - x) = \operatorname{cosec}(x)$	$\sec(\frac{\pi}{2} + x) = -\operatorname{cosec}(x)$	$\sec(\frac{3\pi}{2} - x) = -\operatorname{cosec}(x)$	$\sec(\frac{3\pi}{2} + x) = \operatorname{cosec}(x)$
$\cot(\frac{\pi}{2} - x) = \tan(x)$	$\cot(\frac{\pi}{2} + x) = -\tan(x)$	$\cot(\frac{3\pi}{2} - x) = \tan(x)$	$\cot(\frac{3\pi}{2} + x) = -\tan(x)$

→ Two new identities

- $\sin^2(x) + \cos^2(x) = 1$
- $1 + \tan^2(x) = \sec^2(x) \quad (\cos(x) \neq 0)$
- $1 + \cot^2(x) = \operatorname{cosec}^2(x) \quad (\sin(x) \neq 0)$

- e.g. Simplify $(\sec(\theta)-\cos(\theta))(\operatorname{cosec}(\theta)-\sin(\theta))$.

$\rightarrow \left(\frac{1}{\cos(\theta)}-\cos(\theta)\right)\left(\frac{1}{\sin(\theta)}-\sin(\theta)\right)$

$= \left(\frac{1-\cos^2(\theta)}{\cos(\theta)}\right)\left(\frac{1-\sin^2(\theta)}{\sin(\theta)}\right)$

$= \frac{\sin^2(\theta)}{\cos(\theta)} \times \frac{\cos^2(\theta)}{\sin(\theta)}$

$= \sin(\theta)\cos(\theta) = \boxed{\frac{1}{2}\sin(2\theta)}$ (double angle formula)

Exercise 3B Compound and double angle formulae

→ Compound angle formulae

- Sine "sin cos" "cos sin"

$\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$

$\sin(a-b) = \sin(a)\cos(b) - \cos(a)\sin(b)$

- Cosine "cos cos" "sin sin"

$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$

$\cos(a-b) = \cos(a)\cos(b) + \sin(a)\sin(b)$

- Tangent

$\tan(a+b) = \frac{\tan(a)+\tan(b)}{1-\tan(a)\tan(b)}$

$\tan(a-b) = \frac{\tan(a)-\tan(b)}{1+\tan(a)\tan(b)}$

Calc command
Menu → Algebra(3) → Trig(B)
→ Expand(1) → tExpand( )

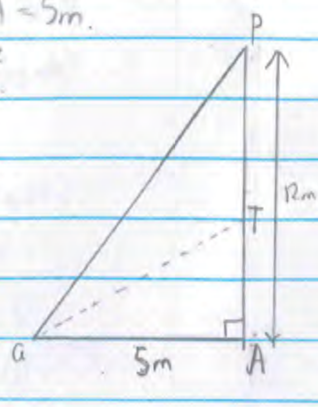
→ Double angle formulae

- $\cos(2ax) = \cos^2(ax) - \sin^2(ax)$
$= 1 - 2\sin^2(ax)$
$= 2\cos^2(ax) - 1$
- $\sin(2ax) = 2\sin(ax)\cos(ax)$
- $\tan(2ax) = \frac{2\tan(ax)}{1-\tan^2(ax)}$
- $\sin^2(ax) = \frac{1}{2}(1-\cos(2ax))$
- $\cos^2(ax) = \frac{1}{2}(1+\cos(2ax))$

→ Applications

- e.g. 1. In a right-angled triangle GAP, AP = 12m and GA = 5m.
The point T on AP is such that $\angle AGT = \angle TGP = x^c$.
Find the exact values of:

(a) $\tan(2x)$ (b) $\tan(x)$ (c) AT



(a) Since $\angle AGT = \angle TGP = x$,
$\triangle AGP = 2x$
$\therefore \tan(2x) = \boxed{\frac{12}{5}}$

(b) $\tan(2x) = \frac{2\tan(x)}{1-\tan^2(x)}$

Let $a = \tan(x)$. $\rightarrow \frac{2a}{1-a^2} = \frac{12}{5}$

$12 - 12a^2 = 10a \rightarrow 12a^2 + 10a - 12 = 0$

$2(3a-2)(2a+3) = 0 \rightarrow a = -\frac{3}{2}$ (reject) or $a = \frac{2}{3}$ (accept) (since $x \in (0, \frac{\pi}{2})$)

$\therefore \tan(x) = \boxed{\frac{2}{3}}$

(c) $\tan(x) = \frac{AT}{5} = \frac{2}{3} \rightarrow 3AT = 10 \rightarrow AT = \boxed{3\frac{1}{3}m}$

· e.g. 2. Let $\cot(\theta) = -\sqrt{5}$ where $\theta \in (\frac{3\pi}{2}, 2\pi)$.

Find $\sec(\theta)$. Hence find $\sin(\frac{\theta}{2})$.

$\rightarrow$ If $\cot(\theta) = -\sqrt{5}$, then $\tan(\theta) = -\frac{\sqrt{5}}{5}$

$\sec^2(\theta) = 1 + \tan^2(\theta) = 1 + (-\frac{\sqrt{5}}{5})^2$

$\sec^2(\theta) = \frac{6}{5} \rightarrow \sec(\theta) = \pm\sqrt{\frac{6}{5}} = \pm\frac{\sqrt{30}}{5}$

Since $\theta \in (\frac{3\pi}{2}, 2\pi)$, $\sec(\theta)$ is positive $\rightarrow \sec(\theta) = \boxed{\frac{\sqrt{30}}{5}}$

$\rightarrow$ Let $A = \frac{\theta}{2}$. Then $\cos(\theta) = \cos(2A) = 1 - 2\sin^2(A)$

If $\sec(\theta) = \frac{\sqrt{30}}{5}$, $\cos(\theta) = \frac{5}{\sqrt{30}} = \frac{\sqrt{30}}{6}$

$\therefore \frac{\sqrt{30}}{6} = 1 - 2\sin^2(A)$

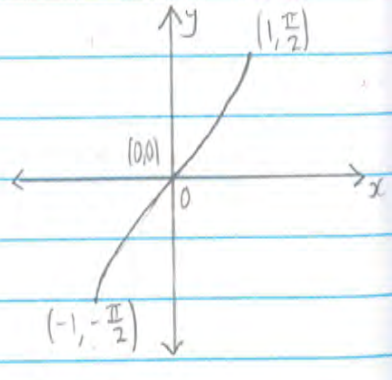
$\sin^2(A) = \frac{1}{2}(1 - \frac{\sqrt{30}}{6}) = \frac{6-\sqrt{30}}{12}$

$\therefore \sin(\frac{\theta}{2}) = \pm\sqrt{\frac{6-\sqrt{30}}{12}}$

Since $\frac{\theta}{2} \in (\frac{3\pi}{4}, \pi)$, $\sin(\frac{\theta}{2})$ is positive $\rightarrow \sin(\frac{\theta}{2}) = \boxed{\sqrt{\frac{6-\sqrt{30}}{12}}}$

Exercise 3C Inverses of circular functions

$\rightarrow$ Inverse sine: $y = \sin^{-1}(x) = \arcsin(x)$



· Domain: $[-1, 1]$

· Range: $[-\frac{\pi}{2}, \frac{\pi}{2}]$ (i.e. Quad 1+4)

· Meaning: $x = \sin(y)$, $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

· Composition: $\sin(\sin^{-1}(x)) = x$ for all $x \in [-1, 1]$

$\sin^{-1}(\sin(x)) = x$ for all $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

· Graphical transformations

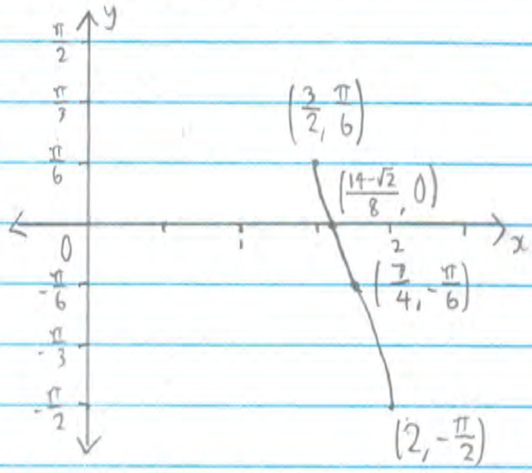
- Swap x and y and then either

① rotate by $90°$ anti-clockwise and then reflect in y-axis; or

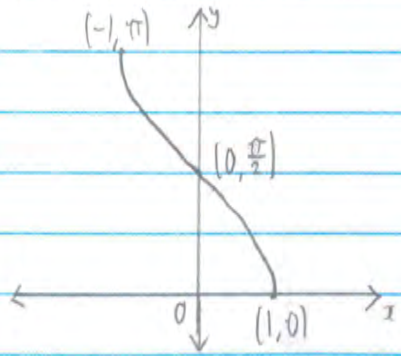
② reflect about line $y = x$

- e.g. $y = \frac{2}{3}\arcsin(7-4x) - \frac{\pi}{6}$
  - → Domain: Solve $-1 \le 7-4x \le 1 \rightarrow \boxed{\frac{3}{2} \le x \le 2}$
    - $7-4x \le 1 \rightarrow -4x \le -6 \rightarrow x \ge \frac{3}{2}$
    - $7-4x \ge -1 \rightarrow -4x \ge -8 \rightarrow x \le 2$
  - → Range: Let $7-4x = 1 \rightarrow \arcsin(7-4x) = \frac{\pi}{2}$
    - $y = \frac{2}{3}\left(\frac{\pi}{2}\right) - \frac{\pi}{6} = \frac{\pi}{6}$
    - Let $7-4x = -1 \rightarrow \arcsin(7-4x) = -\frac{\pi}{2}$.
    - $y = \frac{2}{3}\left(-\frac{\pi}{2}\right) - \frac{\pi}{6} = -\frac{\pi}{2}$
    - $\therefore$ Range: $\boxed{-\frac{\pi}{2} \le y \le \frac{\pi}{6}}$
  - → Endpoints: $\left(\frac{3}{2}, \frac{\pi}{6}\right), \left(2, -\frac{\pi}{2}\right)$
  - → Point of inflection: $\left(\frac{7}{4}, -\frac{\pi}{6}\right)$
  - → x-intercept: $0 = \frac{2}{3}\arcsin(7-4x) - \frac{\pi}{6}$
    - $\frac{\pi}{4} = \arcsin(7-4x)$
    - $\sin\left(\frac{\pi}{4}\right) = 7-4x$
    - $\frac{\sqrt{2}}{2} = 7-4x$
    - $x = \frac{7}{4} - \frac{\sqrt{2}}{8} \rightarrow \left(\frac{14-\sqrt{2}}{8}, 0\right)$

→ Transformations:
1. Dilation by factor $\frac{2}{3}$ from x-axis and 4 from y-axis
2. Reflection in y-axis
3. Translation of $\frac{7}{4}$ units in +ve direction of x-axis and $\frac{\pi}{6}$ units in -ve direction of y-axis



→ Inverse cosine: $y = \cos^{-1}(x) = \arccos(x)$
- Domain: $[-1, 1]$
- Range: $[0, \pi]$ (i.e. Quad 1 + 2)
- Meaning: $x = \cos(y)$, $0 \le y \le \pi$
- Composition: $\cos(\cos^{-1}(x)) = x$ for all $x \in [-1, 1]$
  - $\cos^{-1}(\cos(x)) = x$ for all $x \in [0, \pi]$
- Graphical transformations
  - Swap x and y and then either
    1. rotate by $90°$ anti-clockwise and then reflect in y-axis; or
    2. reflect about line $y = x$



- e.g. $y = \frac{2\pi}{3} - \frac{1}{2}\arccos(3-5x)$
  - → Domain: Solve $-1 \le 3-5x \le 1 \rightarrow \boxed{\frac{2}{5} \le x \le \frac{4}{5}}$
    - $3-5x \le 1 \rightarrow -5x \le -2 \rightarrow x \ge \frac{2}{5}$
    - $3-5x \ge -1 \rightarrow -5x \ge -4 \rightarrow x \le \frac{4}{5}$
  - → Range: Let $3-5x = 1 \rightarrow \arccos(3-5x) = 0$
    - $y = \frac{2\pi}{3} - \frac{1}{2}(0) = \frac{2\pi}{3}$

Let $3-5x=-1 \to \arccos(3-5x)=\pi$

$y=\frac{2\pi}{3}-\frac{\pi}{2}=\frac{\pi}{6}$

$\therefore$ Range: $\boxed{\frac{\pi}{6}\le y\le \frac{2\pi}{3}}$

$\to$ Endpoints: $\left(\frac{2}{5},\frac{2\pi}{3}\right),\left(\frac{4}{5},\frac{\pi}{6}\right)$

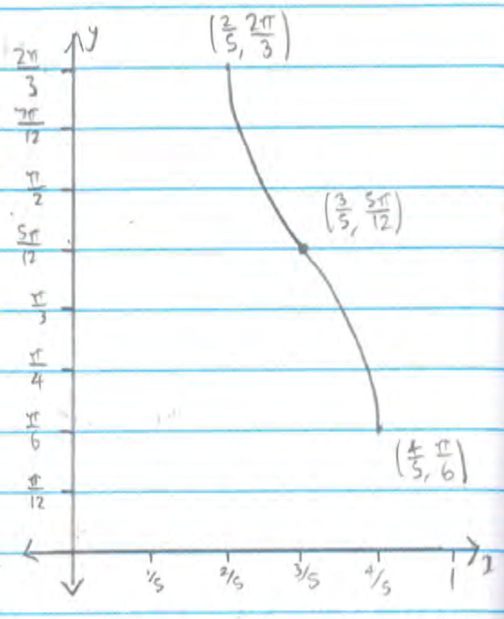
$\to$ Point of inflection: $\left(\frac{3}{5},\frac{5\pi}{12}\right)$

$\to$ Transformations:

① Dilation by factor $\frac{1}{2}$ from x-axis and factor 5 from y-axis

② Reflection in x-axis and y-axis

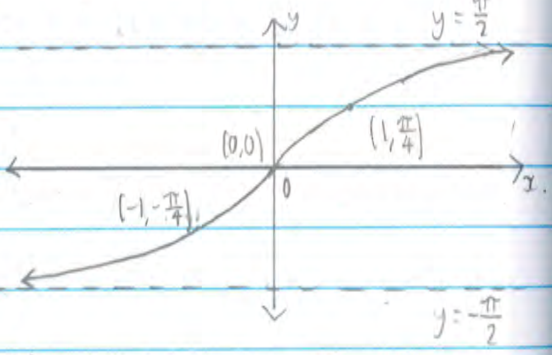
③ Translation of $\frac{3}{5}$ units in +ve direction of x-axis and $\frac{2\pi}{3}$ units in -ve direction of y-axis



$\to$ Inverse tan: $y=\tan^{-1}(x)=\arctan(x)$

- Domain: R
- Range: $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ (ie. Quad 1+4)
- Meaning: $x=\tan(y)$, $-\frac{\pi}{2}<y<\frac{\pi}{2}$
- Composition: $\tan(\tan^{-1}(x))=x$ for all $x\in R$

  $\tan^{-1}(\tan(x))=x$ for all $x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$



- Graphical transformations
  - Swap x and y and then either
    ① rotate by 90° anti-clockwise and then reflect in y-axis, or
    ② reflect about line $y=x$
- e.g. $y=\frac{\pi}{6}-\frac{1}{4}\arctan(2x-\sqrt{3})$

$\to$ Domain: R

$\to$ Range: $\boxed{\frac{\pi}{24}<y<\frac{7\pi}{24}}$

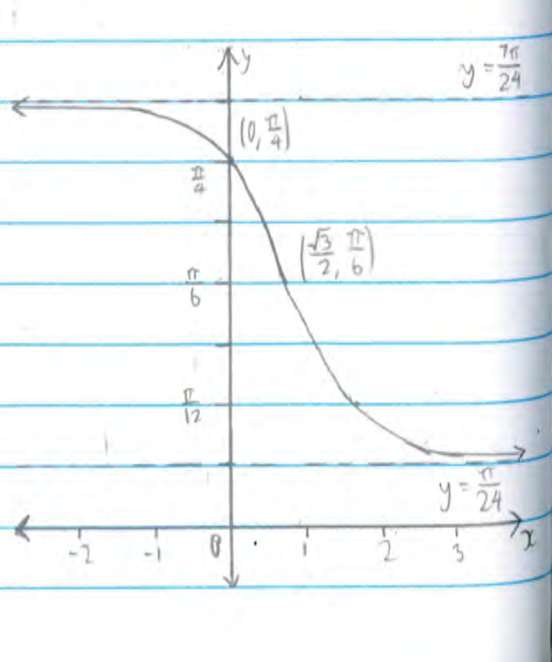
$\cdot\ \frac{\pi}{6}-\frac{1}{4}\left(-\frac{\pi}{2}\right)=\frac{7\pi}{24}$ $\quad\cdot\ \frac{\pi}{6}-\frac{1}{4}\left(\frac{\pi}{2}\right)=\frac{\pi}{24}$

$\to$ Point of inflection: $\left(\frac{\sqrt{3}}{2},\frac{\pi}{6}\right)$

$\to$ y-intercept: $y=\frac{\pi}{6}-\frac{1}{4}\tan^{-1}(-\sqrt{3})$

$=\frac{\pi}{6}-\frac{1}{4}\left(-\frac{\pi}{3}\right)=\frac{\pi}{4}$

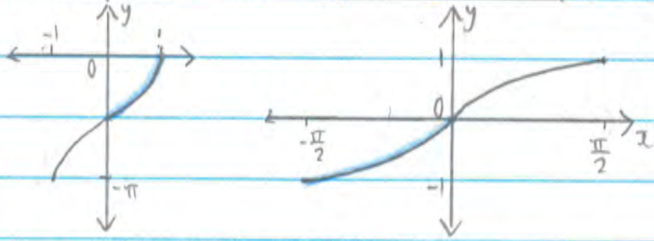
$\to$ No x-intercepts since no value of w such that $\tan^{-1}(w)=\frac{2\pi}{3}$ (from range $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$)



→ Composite functions

- e.g. 1. Find the implied domain and range of $y = \sin(-\cos^{-1}(x))$

$x \rightarrow \boxed{-\cos^{-1}(x)} \longrightarrow \boxed{\sin}$



From above, $\text{ran}(-\cos^{-1}(x)) \not\subseteq \text{dom}(\sin(x))$

So, restrict range of $-\cos^{-1}(x)$ to $[-\frac{\pi}{2}, 0]$ (see blue graph)

$\rightarrow \text{ran}(-\cos^{-1}(x)) \subseteq \text{dom}(\sin(x))$

$\therefore \boxed{\text{dom}(-\cos^{-1}(x)) = [0,1], \ \text{ran}(\sin(x)) = [-1,0]}$

- e.g. 2. Find the implied domain and range of $y = \arcsin\left(\frac{x^2}{1-x^2}\right)$

Let $w = \frac{x^2}{1-x^2} = -1 + \frac{1}{1-x^2}$ so that $y = \arcsin(w)$.

$\rightarrow$ Domain: $-1 \leq w \leq 1$

$w = 1 \rightarrow \frac{x^2}{1-x^2} = 1$

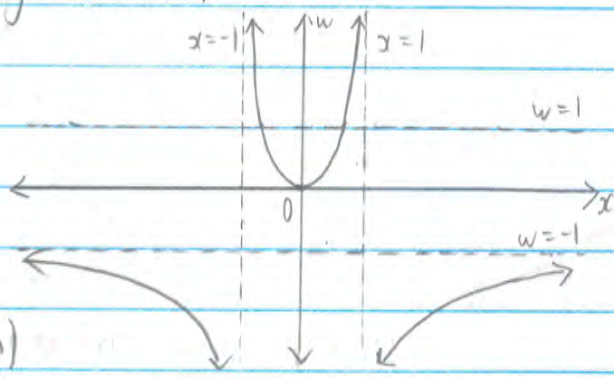
$\rightarrow 2x^2 = 1 \rightarrow x = \pm\frac{\sqrt{2}}{2}$

$\therefore$ Domain: $\boxed{-\frac{\sqrt{2}}{2} \leq x \leq \frac{\sqrt{2}}{2}}$

$\rightarrow$ Range: $\boxed{0 \leq y \leq \frac{\pi}{2}}$

N.B. dom(w): $0 \leq w \leq 1$ (from graph)

Not full set: $-1 \leq w \leq 1$!



- e.g. 3. Find the implied domain and range of $y = -\frac{2}{3}\arccos(3(2^{-x/3}) - \frac{1}{2})$.

Let $w = 3(2^{-x/3}) - \frac{1}{2}$ so that $y = -\frac{2}{3}\arccos(w)$

$\rightarrow$ Domain: $-1 \leq w \leq 1$

$w \neq -1 \rightarrow$ No solns

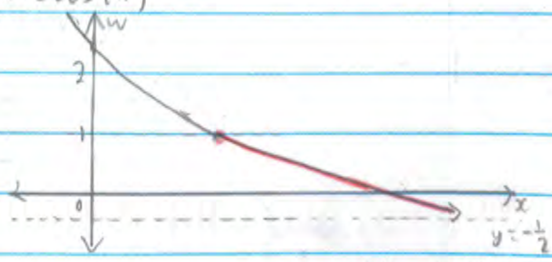
$w = 1 \rightarrow 3(2^{-x/3}) - \frac{1}{2} = 1$

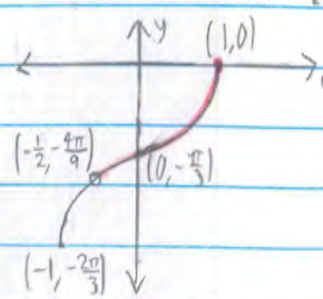
$\rightarrow 2^{-\frac{x}{3}} = \frac{1}{2} = 2^{-1} \rightarrow x = 3$

$\therefore$ Domain: $\boxed{x \geq 3}$

$\rightarrow$ Range: $-\frac{1}{2} < w \leq 1$ for dom of $y = -\frac{2}{3}\cos^{-1}(w)$

$\therefore$ Range: $\boxed{-\frac{4\pi}{9} < y \leq 0}$





Exercise 3D Solutions of equations

→ General solution of trignometric equations

- For $a \in [-1,1]$, $\cos(x) = a \to x = 2n\pi \pm \cos^{-1}(a)$, $n \in \mathbb{Z}$
- For $a \in \mathbb{R}$, $\tan(x) = a \to x = n\pi + \tan^{-1}(a)$, $n \in \mathbb{Z}$
- For $a \in [-1,1]$, $\sin(x) = a \to x = 2n\pi + \sin^{-1}(a)$ or
  $x = (2n+1)\pi - \sin^{-1}(a)$, $n \in \mathbb{Z}$

- e.g. 1. Find the general solution to $\cot(2x - \frac{\pi}{4}) = -1$

$\to \tan(2x - \frac{\pi}{4}) = -1$

In Q1, $2x - \frac{\pi}{4} = \tan^{-1}(1) = \frac{\pi}{4}$

$2x - \frac{\pi}{4} = -\frac{\pi}{4} + n\pi$, $n \in \mathbb{Z}$

$\boxed{x = \frac{n\pi}{2},\ n \in \mathbb{Z}}$

- N.B. Any variant using a Q2 or Q4 angle may also be correct!

- e.g. 2. Find all values of x for which $\sec(2x - \frac{\pi}{3}) = 2$

$\to \cos(2x - \frac{\pi}{3}) = \frac{1}{2}$

In Q1, $2x - \frac{\pi}{3} = \cos^{-1}(\frac{1}{2}) = \frac{\pi}{3}$

$2x - \frac{\pi}{3} = -\frac{\pi}{3} + 2n\pi$ or $2x - \frac{\pi}{3} = \frac{\pi}{3} + 2n\pi$, $n \in \mathbb{Z}$

$2x = 2n\pi$ or $2x = \frac{2\pi}{3} + 2n\pi$, $n \in \mathbb{Z}$

$\therefore \boxed{x = n\pi \text{ or } x = \frac{\pi}{3} + n\pi,\ n \in \mathbb{Z}}$

→ Solution of equations with restricted domain

- e.g. Solve $\operatorname{cosec}(2x - \frac{\pi}{3}) = -\frac{2\sqrt{3}}{3}$ for $x \in [0, 2\pi]$.

$\to \sin(2x - \frac{\pi}{3}) = -\frac{\sqrt{3}}{2}$

If $x \in [0, 2\pi] \to 2x \in [0, 4\pi] \to 2x - \frac{\pi}{3} \in [-\frac{\pi}{3}, \frac{11\pi}{3}]$ (working domain)

In Q1, $2x - \frac{\pi}{3} = \sin^{-1}(\frac{\sqrt{3}}{2}) = \frac{\pi}{3}$

$2x - \frac{\pi}{3} = -\frac{\pi}{3}, \pi + \frac{\pi}{3}, 2\pi - \frac{\pi}{3}, 3\pi + \frac{\pi}{3}, 4\pi - \frac{\pi}{3}$

$= -\frac{\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}, \frac{10\pi}{3}, \frac{11\pi}{3}$

$2x = 0, \frac{5\pi}{3}, 2\pi, \frac{11\pi}{3}, 4\pi$

$\therefore \boxed{x = 0, \frac{5\pi}{6}, \pi, \frac{11\pi}{6}, 2\pi}$

→ Using identities to solve equations

- e.g. 1. Solve $\sin(x) + \cos(x) = 1$ for $x \in [0, 2\pi]$.

Let $x = 2y$. $\to \sin(2y) + \cos(2y) = 1$

$2\sin(y)\cos(y) + (1 - 2\sin^2(y)) = 1$

$*$ $2\sin(y)(\cos(y) - \sin(y)) = 0$

$2\sin(y) = 0$ or $\cos(y) = \sin(y)$ (i.e. $\tan(y) = 1$)

Working domain: If $x \in [0, 2\pi]$, $y = [0, \pi]$.

① $\sin(y) = 0 \rightarrow y = 0, \pi \rightarrow x = 0, 2\pi$

② $\tan(y) = 1 \rightarrow y = \frac{\pi}{4} \rightarrow x = \frac{\pi}{2}$

$\therefore \boxed{x = 0, \frac{\pi}{2}, 2\pi}$

* **Beware**: Don't cancel $\sin(y)$ terms as this loses solutions! It is alright to cancel numerator and denominator terms involving x, but NOT on b.s. of eqn!

e.g. 2. Solve $\cot(x) + 3\tan(x) = 5\operatorname{cosec}(x)$ for $x \in [0, 2\pi]$.

$\rightarrow \frac{\cos(x)}{\sin(x)} + \frac{3\sin(x)}{\cos(x)} = \frac{5}{\sin(x)} \rightarrow \cos^2(x) + 3\sin^2(x) = 5\cos(x)$

$\cos^2(x) + \sin^2(x) + 2\sin^2(x) = 5\cos(x)$

$1 + 2(1 - \cos^2(x)) = 5\cos(x)$

$2\cos^2(x) + 5\cos(x) - 3 = 0$

$(2\cos(x) - 1)(\cos(x) + 3) = 0$

$\therefore \cos(x) = \frac{1}{2}$ (accept) or $\cos(x) = -3$ (reject) (since outside range $[-1, 1]$)

$\therefore$ If $\cos(x) = \frac{1}{2}$, $\boxed{x = \frac{\pi}{3}, \frac{5\pi}{3}}$

e.g. 3. Solve $\sin(8x) = \cos(4x)$ for $x \in [0, 2\pi]$

$\rightarrow 2\sin(4x)\cos(4x) = \cos(4x)$ (double angle formula)

$\cos(4x)(2\sin(4x) - 1) = 0$

$\therefore$ Working domain: If $x \in [0, 2\pi]$, $4x \in [0, 8\pi]$.

① $\cos(4x) = 0$

In Q1, $\cos^{-1}(0) = \frac{\pi}{2}$

$4x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \frac{9\pi}{2}, \frac{11\pi}{2}, \frac{13\pi}{2}, \frac{15\pi}{2}$

$\therefore \boxed{x = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}, \frac{9\pi}{8}, \frac{11\pi}{8}, \frac{13\pi}{8}, \frac{15\pi}{8}}$

② $\sin(4x) = \frac{1}{2}$

In Q1, $\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$

$4x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}, \frac{25\pi}{6}, \frac{29\pi}{6}, \frac{37\pi}{6}, \frac{41\pi}{6}$

$\therefore \boxed{x = \frac{\pi}{24}, \frac{5\pi}{24}, \frac{13\pi}{24}, \frac{17\pi}{24}, \frac{25\pi}{24}, \frac{29\pi}{24}, \frac{37\pi}{24}, \frac{41\pi}{24}}$

$\therefore \boxed{x = \frac{\pi}{24}, \frac{\pi}{8}, \frac{5\pi}{24}, \frac{3\pi}{8}, \frac{13\pi}{24}, \frac{5\pi}{8}, \frac{17\pi}{24}, \frac{7\pi}{8}, \frac{25\pi}{24}, \frac{9\pi}{8}, \frac{29\pi}{24}, \frac{11\pi}{8}, \frac{37\pi}{24}, \frac{13\pi}{8}, \frac{41\pi}{24}, \frac{15\pi}{8}}$

$\rightarrow$ Maximum and minimum values

e.g. Find the max and min values of:

(a) $\cos^2(2x) + 2\cos(2x) + 6$

$= (\cos(2x) + 1)^2 + 5$

$\cos(2x) \in [-1, 1]$

$\cos(2x) + 1 \in [0, 2]$

$(\cos(2x) + 1)^2 \in [0, 4]$

$(\cos(2x) + 1)^2 + 5 \in [5, 9]$

$\boxed{\text{Min: } 5}$

$\boxed{\text{Max: } 9}$

(b) $\frac{1}{\cos^2(2x) + 2\cos(2x) + 6}$

$\boxed{\text{Max: } \frac{1}{5}}$

$\boxed{\text{Min: } \frac{1}{9}}$

→ Applications

e.g. 1. A string is wound around a disc and a horizontal length of string AB is 20cm long. The radius of the disc is 10 cm. The string is then moved so that the end of the string, B', is moved to a point at the same level as O, centre of circle. The line B'P is tangent to the circle.

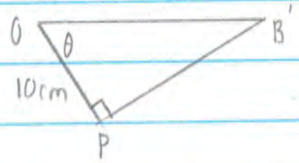
(a) Show that θ satisfies equation $\frac{\pi}{2} - \theta + \tan(\theta) = 2$

→ Consider sector OAP.

$\therefore AP = r\alpha = 10(\frac{\pi}{2} - \theta)$

(since $\angle AOP + \angle POB' = \frac{\pi}{2}$)

→ Consider $\triangle OPB'$.



$\therefore \tan(\theta) = \frac{PB'}{10} = \frac{AB - AP}{10}$

$= \frac{20 - 10(\frac{\pi}{2} - \theta)}{10} = 2 - \frac{\pi}{2} + \theta$

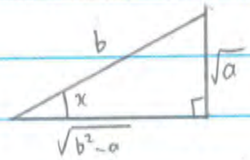
→ $\boxed{\frac{\pi}{2} - \theta + \tan(\theta) = 2}$ (as required)

(b) Find value of θ, correct to two decimal places, which satisfies this eqn.

Solve for $\theta \in (0, \frac{\pi}{2})$, $\boxed{\theta = 0.94}$

e.g. 2. Let $\sin(x) = \frac{\sqrt{a}}{b}$ where $\frac{\pi}{2} < x < \pi$ and $0 < a < b^2$.

Find in simplest form an expression for $\cot(2x)$ in terms of a and b.



→ $|\tan(x)| = \frac{\sqrt{a}}{\sqrt{b^2 - a}}$

$\tan(x) = \frac{-\sqrt{a}}{\sqrt{b^2 - a}}$ (since $\frac{\pi}{2} < x < \pi$)

$\tan(2x) = \frac{2\tan(x)}{1 - \tan^2(x)} = \frac{\frac{-2\sqrt{a}}{\sqrt{b^2 - a}}}{1 - \frac{a}{b^2 - a}}$

$= \frac{2\sqrt{a}\sqrt{b^2 - a}}{2a - b^2}$

$\cot(2x) = \boxed{\frac{2a - b^2}{2\sqrt{a} \cdot \sqrt{b^2 - a}}}$

VCAA 2012 Exam 1 Q10

Question 10

Consider the functions with rules $f(x) = \arcsin\left(\frac{x}{2}\right) + \frac{3}{\sqrt{25x^2 - 1}}$ and $g(x) = \arcsin(3x) - \frac{3}{\sqrt{25x^2 - 1}}$.

a. **i.** Find the maximal domain of $f_1(x) = \arcsin\left(\frac{x}{2}\right)$.

$-1 \le \frac{x}{2} \le 1 \rightarrow \boxed{-2 \le x \le 2}$

ii. Find the maximal domain of $f_2(x) = \frac{3}{\sqrt{25x^2 - 1}}$.

$25x^2 - 1 > 0 \rightarrow x^2 > \frac{1}{25}$

$\therefore \boxed{x \in \left(-\infty, -\frac{1}{5}\right) \cup \left(\frac{1}{5}, \infty\right)}$

iii. Find the largest set of values of $x \in R$ for which $f(x)$ is defined.

$x \in \left[-2, -\frac{1}{5}\right) \cup \left(\frac{1}{5}, 2\right]$

1 + 1 + 1 = 3 marks

b. Given that $h(x) = f(x) + g(x)$ and that $\theta = h\left(\frac{1}{4}\right)$, evaluate $\sin(\theta)$.

Give your answer in the form $\frac{a\sqrt{b}}{c}$, $a, b, c \in Z$. (3 marks)

$h(x) = \arcsin\left(\frac{x}{2}\right) + \frac{3}{\sqrt{25x^2-1}} + \arcsin(3x) - \frac{3}{\sqrt{25x^2-1}}$

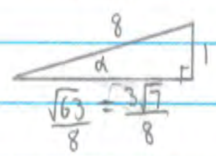
$= \arcsin\left(\frac{x}{2}\right) + \arcsin(3x)$

$\theta = h\left(\frac{1}{4}\right) = \arcsin\left(\frac{1}{8}\right) + \arcsin\left(\frac{3}{4}\right)$

Let $\alpha = \arcsin\left(\frac{1}{8}\right)$ and $\beta = \arcsin\left(\frac{3}{4}\right)$ $\rightarrow \theta = \alpha + \beta$

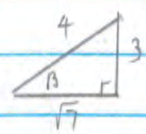
① $\sin(\alpha) = \frac{1}{8}$

$\rightarrow \cos(\alpha) = \frac{\sqrt{63}}{8} = \frac{3\sqrt{7}}{8}$



② $\sin(\beta) = \frac{3}{4}$

$\cos(\beta) = \frac{\sqrt{7}}{4}$



$\sin(\theta) = \sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$

$= \frac{1}{8}\left(\frac{\sqrt{7}}{4}\right) + \frac{3\sqrt{7}}{8}\left(\frac{3}{4}\right) = \boxed{\frac{5\sqrt{7}}{16}}$

VCAA 2012 Exam 1 Q2 (3 marks)

Find all real solutions of equation $2\cos(x) = \sqrt{3}\cot(x)$

$\rightarrow 2\cos(x) = \frac{\sqrt{3}\cos(x)}{\sin(x)} \rightarrow \sqrt{3}\cos(x) = 2\sin(x)\cos(x)$

$\cos(x)\left(2\sin(x) - \sqrt{3}\right) = 0$

① $\cos(x) = 0$

$\boxed{x = \frac{\pi}{2} + \pi n,\ n \in Z}$

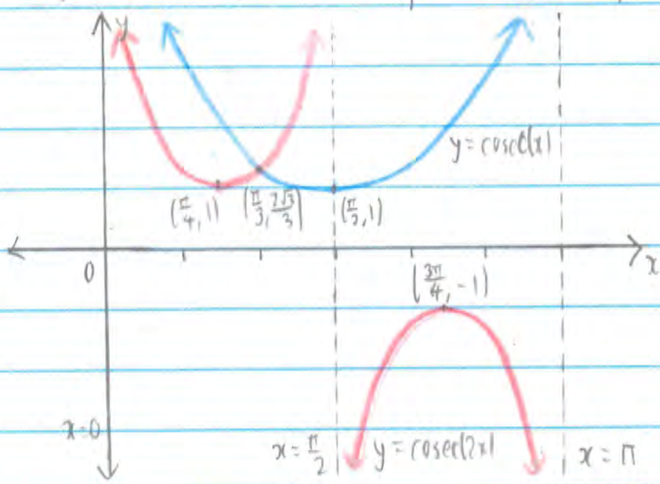
② $\sin(x) = \frac{\sqrt{3}}{2}$

Base angle: $x = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$

$\therefore \boxed{x = \begin{cases} \frac{\pi}{3} + 2\pi n \\ \frac{2\pi}{3} + 2\pi n \end{cases},\ n \in Z}$

VCAA 2015 Exam 1 Q7b (2 marks)

Find $\{x: \operatorname{cosec}(2x) < \operatorname{cosec}(x),\ x \in (0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi)\}$.



Answer: $\boxed{x \in (0, \frac{\pi}{3}) \cup (\frac{\pi}{2}, \pi)}$

VCAA 2016 Exam 1 Q9 (3 marks)

Given than $\cos(x - y) = \frac{3}{5}$ and $\tan(x)\tan(y) = 2$, find $\cos(x+y)$

$\cos(x-y) = \cos(x)\cos(y) + \sin(x)\sin(y) = \frac{3}{5}$

$\tan(x)\tan(y) = \frac{\sin(x)\sin(y)}{\cos(x)\cos(y)} = 2 \rightarrow \sin(x)\sin(y) = 2\cos(x)\cos(y)$

$3\cos(x)\cos(y) = \frac{3}{5} \rightarrow \cos(x)\cos(y) = \frac{1}{5} \rightarrow \sin(x)\sin(y) = \frac{2}{5}$

$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$

$= \frac{1}{5} - \frac{2}{5} = \boxed{-\frac{1}{5}}$

Challenge Question (Exam 1 Type)

Consider the functions with rules $f(x) = \frac{x^2+1}{2x^2-1}$ and $g(x) = \pi - \cos^{-1}(x)$ where each function is defined over its maximum domain.

(a) State the equations of all asymptotes of $f(x)$. (1 mark)

(b) Sketch a graph of $y = f(x)$. Label all asymptotes with their equation and all stationary points with their coordinates (3 marks)

(c) Hence, find the implied domain of $g(f(x))$. (3 marks)

(d) Find the implied range of $g(f(x))$. (2 marks)

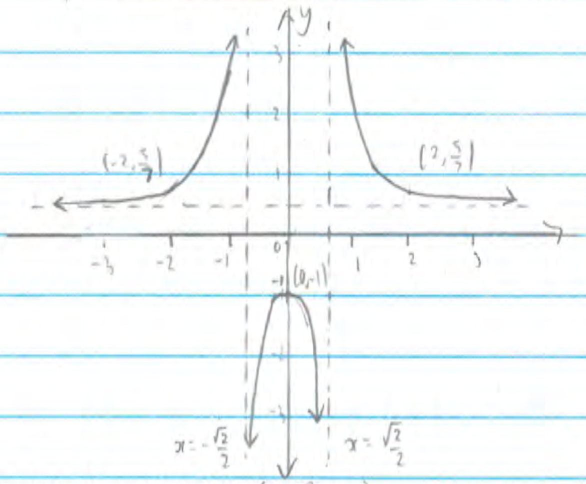
(a) Vertical asymptotes: Solve $2x^2-1=0 \rightarrow x = \pm\frac{\sqrt{2}}{2}$

Oblique asymptotes: $f(x) = \frac{x^2-\frac{1}{2}+\frac{3}{2}}{2x^2-1} = \frac{x^2-\frac{1}{2}}{2x^2-1} + \frac{3}{2(2x^2-1)} = \frac{3}{2(2x^2-1)} + \frac{1}{2}$

$\lim_{x\to\pm\infty}\left(\frac{3}{2(2x^2-1)} + \frac{1}{2}\right) = +\frac{1}{2} = \frac{1}{2} \rightarrow y = \frac{1}{2}$

$\therefore \boxed{x = \pm\frac{\sqrt{2}}{2},\ y = \frac{1}{2}}$

(b)



Turning point: $(0,-1)$

$(f'(x) = 0)$

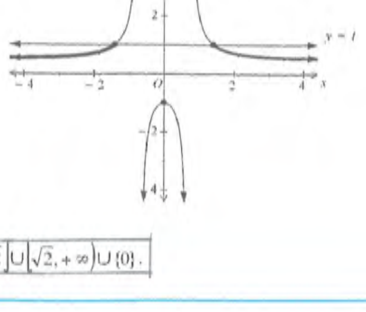
(c) $g(f(x)) = \pi - \cos^{-1}\left(\frac{x^2+1}{2x^2-1}\right)$

Solution: $-1 \le \frac{x^2+1}{2x^2-1} \le 1$

Solve $\frac{x^2+1}{2x^2-1} = 1 \rightarrow x^2+1 = 2x^2-1 \rightarrow x^2-2=0 \rightarrow x = \pm\sqrt{2}$

Solve $\frac{x^2+1}{2x^2-1} = -1 \rightarrow x = 0$

It can therefore be seen from the graph in **part a.** that the solution to $-1 \le \frac{x^2+1}{2x^2-1} \le 1$ is $x \le -\sqrt{2} \cup x \ge \sqrt{2} \cup x = 0$:



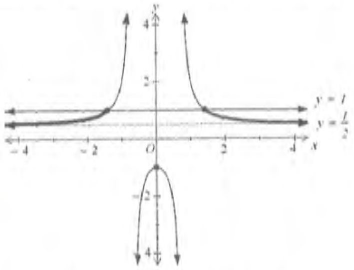
Answer: $(-\infty, -\sqrt{2}] \cup [\sqrt{2}, +\infty) \cup \{0\}$.

(d) Find the implied range of $g(f(x))$.

$f(x) = \frac{x^2+1}{2x^2-1}$ and so $g(f(x)) = \pi - \cos^{-1}\left(\frac{x^2+1}{2x^2-1}\right)$.

• Consider the values of $f(x)$ for $x \in [\sqrt{2}, +\infty)$.

From the graph in **part a.** it can be seen that $f(x) = \frac{x^2+1}{2x^2-1}$ is strictly decreasing for $x \in [\sqrt{2}, +\infty)$:



Therefore $\frac{1}{2} < f(x) \le 1$ for $x \in [\sqrt{2}, +\infty)$.

$g\left(\frac{1}{2}\right) = \pi - \cos^{-1}\left(\frac{1}{2}\right) = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$.

$g(1) = \pi - \cos^{-1}(1) = \pi - 0 = \pi$.

Therefore $\frac{2\pi}{3} < g(f(x)) \le \pi$ for $x \in [\sqrt{2}, +\infty)$. **[H1]**

Consequential on answer to part c.

• Consider the value of $f(x)$ for $x \in (-\infty, -\sqrt{2}]$.

Option 1: From the graph in **part a.** it can be seen that $f(x) = \frac{x^2+1}{2x^2-1}$ is strictly increasing for $x \in (-\infty, -\sqrt{2}]$.

Therefore $\frac{1}{2} < f(x) \le 1$ for $x \in (-\infty, -\sqrt{2}]$.

It follows from the previous calculations that $\frac{2\pi}{3} < g(f(x)) \le \pi$ for $x \in (-\infty, -\sqrt{2}]$.

Option 2: Since $f(x)$ is an even function, it follows from symmetry that $\frac{2\pi}{3} < g(f(x)) \le \pi$.

$f(0) = -1$, $g(-1) = \pi - \cos^{-1}(-1) = 0$

$\rightarrow$ Range: $\boxed{\left(\frac{2\pi}{3}, \pi\right] \cup \{0\}}$