4.3

SYSTEMS OF LINEAR EQUATIONS IN THREE VARIABLES

The techniques that you learned in Section 4.2 can be extended to systems of equations in more than two variables. In this section we use elimination of variables to solve systems of equations in three variables.

Definition

The equation $5x - 4y = 7$ is called a linear equation in two variables because its graph is a straight line. The equation $2x + 3y - 4z = 12$ is similar in form, and so it is a linear equation in three variables. An equation in three variables is graphed in a three-dimensional coordinate system. The graph of a linear equation in three variables is a plane, not a line. We will not graph equations in three variables in this text, but we can solve systems without graphing. In general, we make the following definition.

> **Linear Equation in Three Variables**
>
> If A, B, C, and D are real numbers, with A, B, and C not all zero, then
>
> $$Ax + By + Cz = D$$
>
> is called a **linear equation in three variables.**

Solving a System by Elimination

A solution to an equation in three variables is an **ordered triple** such as $(-2, 1, 5)$, where the first coordinate is the value of x, the second coordinate is the value of y, and the third coordinate is the value of z. There are infinitely many solutions to a linear equation in three variables.

The solution to a system of equations in three variables is the set of all ordered triples that satisfy all of the equations of the system. The techniques for solving a system of linear equations in three variables are similar to those used on systems of linear equations in two variables. We eliminate variables by either substitution or addition.

study tip

Everyone knows that you must practice to be successful with musical instruments, foreign languages, and sports. Success in algebra also requires regular practice. Thus budget your time so that you have a regular practice period for algebra.

EXAMPLE 1

A linear system with a single solution

Solve the system:

$$(1) \qquad x + y - z = -1$$
$$(2) \qquad 2x - 2y + 3z = 8$$
$$(3) \qquad 2x - y + 2z = 9$$

Solution

We can eliminate z from Eqs. (1) and (2) by multiplying Eq. (1) by 3 and adding it to Eq. (2):

$$3x + 3y - 3z = -3 \qquad \text{Eq. (1) multiplied by 3}$$
$$2x - 2y + 3z = 8 \qquad \text{Eq. (2)}$$
$$(4) \qquad 5x + y \qquad = 5$$

Now we must eliminate the same variable, z, from another pair of equations. Eliminate z from (1) and (3):

$$2x + 2y - 2z = -2 \quad \text{Eq. (1) multiplied by 2}$$
$$\underline{2x - y + 2z = 9} \quad \text{Eq. (3)}$$
$$(5) \quad 4x + y = 7$$

Equations (4) and (5) give us a system with two variables. We now solve this system. Eliminate y by multiplying Eq. (5) by -1 and adding the equations:

$$5x + y = 5 \quad \text{Eq. (4)}$$
$$\underline{-4x - y = -7} \quad \text{Eq. (5) multiplied by } -1$$
$$x = -2$$

Now that we have x, we can replace x by -2 in Eq. (5) to find y:

$$4x + y = 7$$
$$4(-2) + y = 7$$
$$-8 + y = 7$$
$$y = 15$$

Now replace x by -2 and y by 15 in Eq. (1) to find z:

$$x + y - z = -1$$
$$-2 + 15 - z = -1$$
$$13 - z = -1$$
$$-z = -14$$
$$z = 14$$

Check that $(-2, 15, 14)$ satisfies all three of the original equations. The solution set is $\{(-2, 15, 14)\}$. ◼

The strategy that we follow for solving a system of three linear equations in three variables is stated as follows.

Solving a System in Three Variables

1. Use substitution or addition to eliminate any one of the variables from a pair of equations of the system. Look for the easiest variable to eliminate.

2. Eliminate the same variable from another pair of equations of the system.

3. Solve the resulting system of two equations in two unknowns.

4. After you have found the values of two of the variables, substitute into one of the original equations to find the value of the third variable.

5. Check the three values in all of the original equations.

In the next example we use a combination of addition and substitution.

E X A M P L E   2   **Using addition and substitution**

Solve the system:

$$(1) \quad x + y = 4$$
$$(2) \quad 2x - 3z = 14$$
$$(3) \quad 2y + z = 2$$

Solution

h e l p f u l / h i n t

In Example 2 we chose to eliminate y first. Try solving Example 2 by first eliminating z. Write $z = 2 - 2y$ and then substitute $2 - 2y$ for z in Eqs. (1) and (2).

From Eq. (1) we get $y = 4 - x$. If we substitute $y = 4 - x$ into Eq. (3), then Eqs. (2) and (3) will be equations involving x and z only.

$$(3) \qquad 2y + z = 2$$
$$2(4 - x) + z = 2 \quad \text{Replace } y \text{ by } 4 - x.$$
$$8 - 2x + z = 2 \quad \text{Simplify.}$$
$$(4) \qquad -2x + z = -6$$

Now solve the system consisting of Eqs. (2) and (4) by addition:

$$
\begin{array}{rl}
2x - 3z = 14 & \text{Eq. (2)} \\
\underline{-2x + \ z = -6} & \text{Eq. (4)} \\
-2z = 8 & \\
z = -4 &
\end{array}
$$

Use Eq. (3) to find y:

$$
\begin{array}{rl}
2y + z = 2 & \text{Eq. (3)} \\
2y + (-4) = 2 & \text{Let } z = -4. \\
2y = 6 & \\
y = 3 &
\end{array}
$$

Use Eq. (1) to find x:

$$
\begin{array}{rl}
x + y = 4 & \text{Eq. (1)} \\
x + 3 = 4 & \text{Let } y = 3. \\
x = 1 &
\end{array}
$$

Check that $(1, 3, -4)$ satisfies all three of the original equations. The solution set is $\{(1, 3, -4)\}$. ■

s t u d y \ t i p

When one student asks another for help, often the one who does the explaining is the one who learns the most. When you work together, don't let one person do all of the talking. If you must work alone, then try explaining things to yourself. You might even try talking to yourself.

C A U T I O N   In solving a system in three variables it is essential to keep your work organized and neat. Writing short notes that explain your steps (as was done in the examples) will allow you to go back and check your work.

Graphs of Equations in Three Variables

The graph of any equation in three variables can be drawn on a three-dimensional coordinate system. The graph of a linear equation in three variables is a plane. To solve a system of three linear equations in three variables by graphing, we would have to draw the three planes and then identify the points that lie on all three of them. This method would be difficult even when the points have simple coordinates. So we will not attempt to solve these systems by graphing.

By considering how three planes might intersect, we can better understand the different types of solutions to a system of three equations in three variables. Figure 4.6, on the next page, shows some of the possibilities for the positioning of three planes in three-dimensional space. In most of the problems that we will solve the planes intersect at a single point as in Fig. 4.6(a). The solution set consists of one ordered triple. However, the system may include two equations corresponding to parallel planes that have no intersection. In this case the equations are said to be **inconsistent.** If the system has at least two inconsistent equations, then the solution set is the empty set [see Figs. 4.6(b) and 4.6(c)].

There are two ways in which the intersection of three planes can consist of infinitely many points. The intersection could be a line or a plane. To get a line, we can

have either three different planes intersecting along a line, as in Fig. 4.6(d) or two equations for the same plane, with the third plane intersecting that plane. If all three equations are equations of the same plane, we get that plane for the intersection. We will not solve systems corresponding to all of the possible configurations described. The following examples illustrate two of these cases.
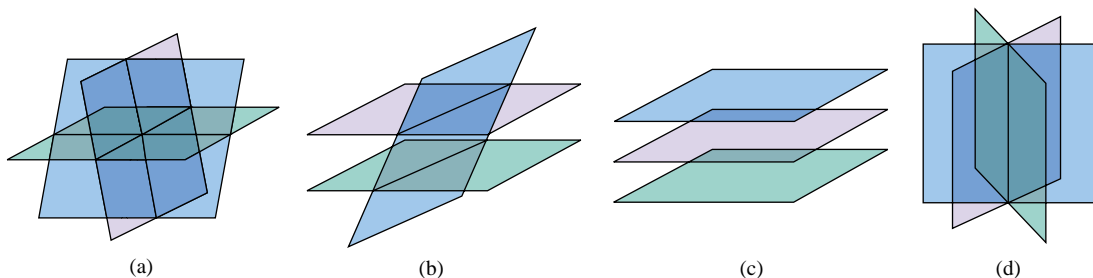

(a)   (b)   (c)   (d)

F I G U R E   4 . 6

E X A M P L E   3

An inconsistent system of three linear equations

Solve the system:

$$
\begin{aligned}
(1) \quad & x + y - z = 5 \\
(2) \quad & 3x - 2y + z = 8 \\
(3) \quad & 2x + 2y - 2z = 7
\end{aligned}
$$

Solution

We can eliminate the variable z from Eqs. (1) and (2) by adding them:

$$
\begin{aligned}
(1) \quad & x + y - z = 5 \\
(2) \quad & \underline{3x - 2y + z = 8} \\
& 4x - y = 13
\end{aligned}
$$

To eliminate z from Eqs. (1) and (3), multiply Eq. (1) by -2 and add the resulting equation to Eq. (3):

$$
\begin{aligned}
-2x - 2y + 2z &= -10 \quad \text{Eq. (1) multiplied by } -2 \\
\underline{2x + 2y - 2z} &= \underline{7} \quad \text{Eq. (3)} \\
0 &= -3
\end{aligned}
$$

Because the last equation is false, there are two inconsistent equations in the system. Therefore the solution set is the empty set. ∎

E X A M P L E   4

A dependent system of three equations

Solve the system:

$$
\begin{aligned}
(1) \quad & 2x - 3y - z = 4 \\
(2) \quad & -6x + 9y + 3z = -12 \\
(3) \quad & 4x - 6y - 2z = 8
\end{aligned}
$$

helpful hint

If you recognize that multiplying Eq. (1) by -3 will produce Eq. (2), and multiplying Eq. (1) by 2 will produce Eq. (3), then you can conclude that all three equations are equivalent and there is no need to add the equations.

Solution

We will first eliminate x from Eqs. (1) and (2). Multiply Eq. (1) by 3 and add the resulting equation to Eq. (2):

$$
\begin{aligned}
6x - 9y - 3z &= 12 \quad \text{Eq. (1) multiplied by 3} \\
\underline{-6x + 9y + 3z} &= \underline{-12} \quad \text{Eq. (2)} \\
0 &= 0
\end{aligned}
$$

The last statement is an identity. The identity occurred because Eq. (2) is a multiple of Eq. (1). In fact, Eq. (3) is also a multiple of Eq. (1). These equations are dependent. They are all equations for the same plane. The solution set is the set of all points on that plane,

$$\{(x, y, z) \mid 2x - 3y - z = 4\}.$$

Applications

Problems involving three unknown quantities can often be solved by using a system of three equations in three variables.

E X A M P L E   5

Finding three unknown rents

Theresa took in a total of $1,240 last week from the rental of three condominiums. She had to pay 10% of the rent from the one-bedroom condo for repairs, 20% of the rent from the two-bedroom condo for repairs, and 30% of the rent from the three-bedroom condo for repairs. If the three-bedroom condo rents for twice as much as the one-bedroom condo and her total repair bill was $276, then what is the rent for each condo?

Solution

Let x, y, and z represent the rent on the one-bedroom, two-bedroom, and three-bedroom condos, respectively. We can write one equation for the total rent, another equation for the total repairs, and a third equation expressing the fact that the rent for the three-bedroom condo is twice that for the one-bedroom condo:

$$x + y + z = 1240$$
$$0.1x + 0.2y + 0.3z = 276$$
$$z = 2x$$

Substitute $z = 2x$ into both of the other equations to eliminate z:

$$x + y + 2x = 1240$$
$$0.1x + 0.2y + 0.3(2x) = 276$$

$$3x + \quad y = 1240$$
$$0.7x + 0.2y = 276$$

$$-2(3x + y) = -2(1240) \quad \text{Multiply each side by } -2.$$
$$10(0.7x + 0.2y) = 10(276) \quad \text{Multiply each side by } 10.$$

$$-6x - 2y = -2480$$
$$\underline{7x + 2y = 2760} \quad \text{Add.}$$
$$x \quad\quad = 280$$

$$z = 2(280) = 560 \quad \text{Because } z = 2x$$
$$280 + y + 560 = 1240 \quad \text{Because } x + y + z = 1240$$
$$y = 400$$

Check that (280, 400, 560) satisfies all three of the original equations. The condos rent for $280, $400, and $560 per week.

> **helpful hint**
>
> A problem involving two unknowns can often be solved with one variable as in Chapter 2. Likewise, you can often solve a problem with three unknowns using only two variables. Solve Example 5 by letting a, b, and $2a$ be the rent for a one-bedroom, two-bedroom, and a three-bedroom condo.

True or false? Explain your answer.

1. The point $(1, -2, 3)$ is in the solution set to the equation $x + y - z = 4$.

2. The point $(4, 1, 1)$ is the only solution to the equation $x + y - z = 4$.

3. The ordered triple $(1, -1, 2)$ satisfies $x + y + z = 2$, $x - y - z = 0$, and $2x + y - z = -1$.

4. Substitution cannot be used on three equations in three variables.

5. Two distinct planes are either parallel or intersect in a single point.

6. The equations $x - y + 2z = 6$ and $x - y + 2z = 4$ are inconsistent.

7. The equations $3x + 2y - 6z = 4$ and $-6x - 4y + 12z = -8$ are dependent.

8. The graph of $y = 2x - 3z + 4$ is a straight line.

9. The value of x nickels, y dimes, and z quarters is $0.05x + 0.10y + 0.25z$ cents.

10. If $x = -2$, $z = 3$, and $x + y + z = 6$, then $y = 7$.

4.3   EXERCISES

Reading and Writing   *After reading this section, write out the answers to these questions. Use complete sentences.*

1. What is a linear equation in three variables?

2. What is an ordered triple?

3. What is a solution to a system of linear equations in three variables?

4. How do we solve systems of linear equations in three variables?

5. What does the graph of a linear equation in three variables look like?

6. How are the planes positioned when a system of linear equations in three variables is inconsistent?

Solve each system of equations. See Examples 1 and 2.

7. $\begin{aligned} x + y + z &= 2 \\ x + 2y - z &= 6 \\ 2x + y - z &= 5 \end{aligned}$

8. $\begin{aligned} 2x - y + 3z &= 14 \\ x + y - 2z &= -5 \\ 3x + y - z &= 2 \end{aligned}$

9. $\begin{aligned} x - 2y + 4z &= 3 \\ x + 3y - 2z &= 6 \\ x - 4y + 3z &= -5 \end{aligned}$

10. $\begin{aligned} 2x + 3y + z &= 13 \\ -3x + 2y + z &= -4 \\ 4x - 4y + z &= 5 \end{aligned}$

11. $\begin{aligned} 2x - y + z &= 10 \\ 3x - 2y - 2z &= 7 \\ x - 3y - 2z &= 10 \end{aligned}$

12. $\begin{aligned} x - 3y + 2z &= -11 \\ 2x - 4y + 3z &= -15 \\ 3x - 5y - 4z &= 5 \end{aligned}$

13. $\begin{aligned} 2x - 3y + z &= -9 \\ -2x + y - 3z &= 7 \\ x - y + 2z &= -5 \end{aligned}$

14. $\begin{aligned} 3x - 4y + z &= 19 \\ 2x + 4y + z &= 0 \\ x - 2y + 5z &= 17 \end{aligned}$

15. $\begin{aligned} 2x - 5y + 2z &= 16 \\ 3x + 2y - 3z &= -19 \\ 4x - 3y + 4z &= 18 \end{aligned}$

16. $\begin{aligned} -2x + 3y - 4z &= 3 \\ 3x - 5y + 2z &= 4 \\ -4x + 2y - 3z &= 0 \end{aligned}$

17. $\begin{aligned} x + y &= 4 \\ y - z &= -2 \\ x + y + z &= 9 \end{aligned}$

18. $\begin{aligned} x + y - z &= 0 \\ x - y &= -2 \\ y + z &= 10 \end{aligned}$

19. $x + y \quad\quad = 7$
$\quad\quad y - \; z = -1$
$\quad x \quad\quad + 3z = 18$

20. $2x - y \quad\quad = -8$
$\quad\quad y + 3z = 22$
$\quad x \quad\quad - \; z = -8$

Solve each system. See Examples 3 and 4.

21. $x - \; y + 2z = 3$
$2x + \; y - \; z = 5$
$3x - 3y + 6z = 4$

22. $2x - \; 4y + \; 6z = 12$
$6x - 12y + 18z = 36$
$-x + \; 2y - \; 3z = -6$

23. $\quad 3x - \; y + \; z = 5$
$\quad 9x - 3y + 3z = 15$
$-12x + 4y - 4z = -20$

24. $\quad 4x - 2y - 2z = 5$
$\quad 2x - \; y - \; z = 7$
$-4x + 2y + 2z = 6$

25. $x - y \quad\quad = 3$
$\quad\quad y + \; z = 8$
$2x \quad\quad + 2z = 7$

26. $2x - y \quad\quad = 6$
$\quad\quad 2y + \; z = -4$
$8x \quad\quad + 2z = 3$

27. $0.10x + 0.08y - \; 0.04z = 3$
$\quad 5x + \quad 4y - \quad 2z = 150$
$0.3x + 0.24y - 0.12z = 9$

28. $0.06x - 0.04y + \quad z = 6$
$\quad 3x - \quad 2y + 50z = 300$
$0.03x - 0.02y + 0.5z = 3$

Use a calculator to solve each system.

29. $\quad 3x + \quad 2y - \; 0.4z = 0.1$
$3.7x - 0.2y + 0.05z = 0.41$
$-2x + 3.8y - \quad 2.1z = -3.26$

30. $\quad 3x - 0.4y + 9z = 1.668$
$0.3x + \quad 5y - 8z = -0.972$
$\quad 5x - \quad 4y - 8z = 1.8$

Solve each problem by using a system of three equations in three unknowns. See Example 5.

31. *Diversification.* Ann invested a total of $12,000 in stocks, bonds, and a mutual fund. She received a 10% return on her stock investment, an 8% return on her bond investment, and a 12% return on her mutual fund. Her total return was $1,230. If the total investment in stocks and bonds equaled her mutual fund investment, then how much did she invest in each?

32. *Paranoia.* Fearful of a bank failure, Norman split his life savings of $60,000 among three banks. He received 5%, 6%, and 7% on the three deposits. In the account earning 7% interest, he deposited twice as much as in the account earning 5% interest. If his total earnings were $3,760, then how much did he deposit in each account?

33. *Big tipper.* On Monday Headley paid $1.70 for two cups of coffee and one doughnut, including the tip. On Tuesday he paid $1.65 for two doughnuts and a cup of coffee, including the tip. On Wednesday he paid $1.30 for one coffee and one doughnut, including the tip. If he always tips the same amount, then what is the amount of each item?

34. *Weighing in.* Anna, Bob, and Chris will not disclose their weights but agree to be weighed in pairs. Anna and Bob together weigh 226 pounds. Bob and Chris together weigh 210 pounds. Anna and Chris together weigh 200 pounds. How much does each student weigh?
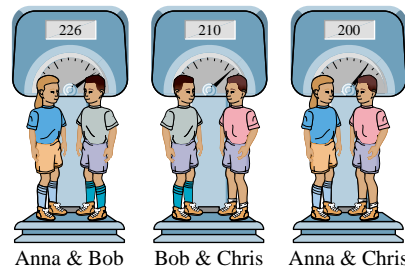
FIGURE FOR EXERCISE 34

35. *Lunch-box special.* Salvador's Fruit Mart sells variety packs. The small pack contains three bananas, two apples, and one orange for $1.80. The medium pack contains four bananas, three apples, and three oranges for $3.05. The family size contains six bananas, five apples, and four oranges for $4.65. What price should Salvador charge for his lunch-box special that consists of one banana, one apple, and one orange?

36. *Three generations.* Edwin, his father, and his grandfather have an average age of 53. One-half of his grandfather's age, plus one-third of his father's age, plus one-fourth of Edwin's age is 65. If 4 years ago, Edwin's grandfather was four times as old as Edwin, then how old are they all now?

37. *Error in the scale.* Alex is using a scale that is known to have a constant error. A can of soup and a can of tuna are placed on this scale, and it reads 24 ounces. Now four identical cans of soup and three identical cans of tuna are placed on an accurate scale, and a weight of 80 ounces is recorded. If two cans of tuna weigh 18 ounces on the bad scale, then what is the amount of error in the scale and what is the correct weight of each type of can?

38. *Three-digit number.* The sum of the digits of a three-digit number is 11. If the digits are reversed, the new number is 46 more than five times the old number. If the hundreds digit plus twice the tens digit is equal to the units digit, then what is the number?

39. *Working overtime.* To make ends meet, Ms. Farnsby works three jobs. Her total income last year was $48,000. Her income from teaching was just $6,000 more than her income from house painting. Royalties from her textbook sales were one-seventh of the total money she received from teaching and house painting. How much did she make from each source last year?

40. *Pocket change.* Harry has $2.25 in nickels, dimes, and quarters. If he had twice as many nickels, half as many dimes, and the same number of quarters, he would have $2.50. If he has 27 coins altogether, then how many of each does he have?

GETTING MORE INVOLVED

41. *Exploration.* Draw diagrams showing the possible ways to position three planes in three-dimensional space.

42. *Discussion.* Make up a system of three linear equations in three variables for which the solution set is $\{(0, 0, 0)\}$. A system with this solution set is called a *homogeneous* system. Why do you think it is given that name?

43. *Cooperative learning.* Working in groups, do parts (a)–(d) below. Then write a report on your findings.
  a) Find values of a, b, and c so that the graph of $y = ax^2 + bx + c$ goes through the points $(-1, -2)$, $(1, 0)$, and $(2, 7)$.
  b) Arbitrarily select three ordered pairs and find the equation of the parabola that goes through the three points.
  c) Could more than one parabola pass through three given points? Give reasons for your answer.
  d) Explain how to pick three points for which no parabola passes through all of them.

4.4 SOLVING LINEAR SYSTEMS USING MATRICES

You solved linear systems in two variables by substitution and addition in Sections 4.1 and 4.2. Those methods are done differently on each system. In this section you will learn the Gaussian elimination method, which is related to the addition method. The Gaussian elimination method is performed in the same way on every system. We first need to introduce some new terminology.

In this section

- Matrices
- The Augmented Matrix
- The Gaussian Elimination Method
- Inconsistent and Dependent Equations

Matrices

A **matrix** is a rectangular array of numbers. The **rows** of a matrix run horizontally, and the **columns** of a matrix run vertically. A matrix with m rows and n columns has **order** $m \times n$ (read "m by n"). Each number in a matrix is called an **element** or **entry** of the matrix.

EXAMPLE 1

Order of a matrix

Determine the order of each matrix.

a) $\begin{bmatrix} -1 & 2 \\ 5 & \sqrt{2} \\ 0 & 3 \end{bmatrix}$ b) $\begin{bmatrix} 2 & 3 \\ -1 & 5 \end{bmatrix}$ c) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ -1 & 0 & 2 \end{bmatrix}$ d) $\begin{bmatrix} 1 & 3 & 6 \end{bmatrix}$

Solution

Because matrix (a) has 3 rows and 2 columns, its order is 3×2. Matrix (b) is a 2×2 matrix, matrix (c) is a 3×3 matrix, and matrix (d) is a 1×3 matrix. ■

The Augmented Matrix

The solution to a system of linear equations such as

$$x - 2y = -5$$
$$3x + y = 6$$

study tip

As soon as possible after class, find a quiet place and work on your homework. The longer you wait, the harder it is to remember what happened in class.