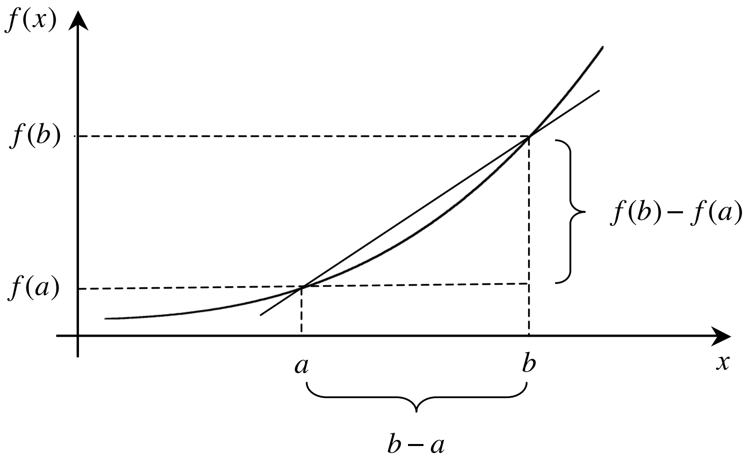
Appendix A

First Principles

The graph below shows how a function $f(x)$ might change between $x = a$ and $x = b$.



The gradient of the chord from $(a, f(a))$ to $(b, f(b)$ is

$$\frac{\Delta y}{\Delta x} = \frac{\text{change in } y}{\text{change in } x} = \frac{f(b) - f(a)}{b - a}.$$

As the width of the interval $[a, b]$ decreases, the approximation

$$\frac{\Delta y}{\Delta x} = \frac{f(b) - f(a)}{b - a}.$$

becomes closer to *the gradient of the tangent line* to the graph of $f(x)$ at $x = a$, and so to *the derivative* of $f(x)$ at $x = a$.

If we put $b = a + h$, then the derivative of $f(x)$ at $x = a$ is given by the limit

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{h \to 0} \frac{f(a + h) - f(a)}{(a + h) - a}$$

Definition

The derivative of $y = f(x)$ at the point $(a, f(a))$ is given by the limit

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Example

first principles at $x = a$

Find the derivative of $y = x^2$ at $x = a$ using first principles

1. From the definition ...

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{(a+h)^2 - a^2}{h}$$

2. Expanding, then simplifying and taking the limit ...

$$\begin{aligned}\frac{dy}{dx} &= \lim_{h \to 0} \frac{(a^2 + 2ah + h^2) - a^2}{h} \\ &= \lim_{h \to 0} \frac{2ah + h^2}{h} \\ &= \lim_{h \to 0} 2a + h \\ &= 2a\end{aligned}$$

The derivative at $x = a$ is $\dfrac{dy}{dx} = 2a$

Example

first principles without using $x = a$

Differentiate $y = x^2 + 4x + 2$ using first principles

1. From the definition ...

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{[(x+h)^2 + 4(x+h) + 2] - [x^2 + 4x + 2]}{h}$$

2. Expanding, then simplifying and taking the limit ...

$$\begin{aligned}\frac{dy}{dx} &= \lim_{h \to 0} \frac{[x^2 + (2h+4)x + (h^2 + 4h + 2)] - [x^2 + 4x + 2]}{h} \\ &= \lim_{h \to 0} \frac{2hx + (h^2 + 4h)}{h} \\ &= \lim_{h \to 0} 2x + h + 4 \\ &= 2x + 4\end{aligned}$$

The derivative is $\dfrac{dy}{dx} = 2x + 4$

Example

cubic function

Differentiate $y = x^3$ using first principles

1. From the definition ...

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h}$$

2. Expanding, then simplifying and taking the limit ...

$$\begin{aligned} \frac{dy}{dx} &= \lim_{h \to 0} \frac{(x^3 + 3x^2h + 3xh^2 + h^3) - x^3}{h} \\ &= \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3}{h} \\ &= \lim_{h \to 0} 3x^2 + 3xh + h^2 \\ &= 3x^2 \end{aligned}$$

The derivative is $\frac{dy}{dx} = 3x^2$

Example

rational function

Differentiate $f(x) = \frac{x+1}{x+2}$ using first principles

1. From the definition ...

$$f'(x) = \lim_{h \to 0} \frac{\frac{(x+h)+1}{(x+h)+2} - \frac{x+1}{x+2}}{h}$$

2. Expanding, then simplifying and taking the limit ...

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{(x+h+1)(x+2) - (x+1)(x+h+2)}{(x+h+2)(x+2)h} \\ &= \lim_{h \to 0} \frac{1}{(x+h+2)(x+2)} \\ &= \frac{1}{(x+2)(x+2)} \\ &= \frac{1}{(x+2)^2} \end{aligned}$$

The derivative is $f'(x) = \frac{1}{(x+2)^2}$.