Section 2 Integrating Polynomials

Integration is a technique for finding, amoungst other things, the area under curves. Conceptually, it is like the method of left and right rectangles, but the number of subintervals that the interval of interest is broken up into is infinite, so we get an exact area where the lower and upper bounds are equal. We will not give the details of how one takes the limit of an infinite number of subintervals - we will just state some integration results.

Integration involves anti derivatives, so we will first look at these. The anti derivative of a function f is another function F such that

$$f(x) = F'(x)$$

Thus if $f(x)$ is the derivative of $F(x)$ then $F(x)$ is the anti derivative of $f(x)$. Worksheet 3.8 has an introduction to derivatives. Therefore we can reverse the rules that we had for polynomial differentiation to get anti derivative rules. Recall that if $f(x) = ax^n$, then $f'(x) = anx^{n-1}$. So if $g(x) = bx^m$ then an anti derivative $G(x)$ (such that $G'(x) = g(x)$) is given by

$$G(x) = \frac{b}{m+1}x^{m+1} \quad m \neq -1$$

We add one to the power of x then divide by the new power of x. Note that if

$$\begin{aligned} G(x) &= \frac{b}{m+1}x^{m+1} \\ \text{then} \quad G'(x) &= \frac{b}{m+1}(m+1)x^{m+1-1} \\ &= bx^m \\ &= g(x) \end{aligned}$$

which is what is required. Given $f'(x) = 2x$, then we could have $f(x) = x^2+1$ or $f(x) = x^2+3$ or $f(x) = x^2-4$; notice they differ by the constant term. To compensate for this - the property that the derivative of a constant is zero - we add a constant, usually denoted as c, to the anti derivative. We need more information to find distinct values of c.

Example 1 : Find the anti derivative $F(x)$ of the function $f(x) = 2x + 1$. Note $x^0 = 1$.

$$\begin{aligned} F(x) &= \frac{2x^{1+1}}{2} + \frac{1x^{0+1}}{1} + c \\ &= x^2 + x + c \end{aligned}$$

Example 2 : Find the anti derivative $G(x)$ of the function $g(x) = x^2 + 3x$.

$$\begin{aligned} G(x) &= \frac{x^{2+1}}{3} + \frac{3x^{1+1}}{2} + c \\ &= \frac{x^3}{3} + \frac{3x^2}{2} + c \end{aligned}$$

Example 3 : Find the anti derivative $H(x)$ of the function $h(x) = 5x^4 + 3x^2 + x + x^{-5} + 3$.

$$\begin{aligned} H(x) &= \frac{5x^{4+1}}{5} + \frac{3x^{2+1}}{3} + \frac{x^{1+1}}{2} + \frac{x^{-5+1}}{-4} + \frac{3x^{0+1}}{1} + c \\ &= x^5 + x^3 + \frac{x^2}{2} - \frac{x^{-4}}{4} + 3x + c \end{aligned}$$

Example 4 : Find the anti derivative $F(x)$ of $f(x) = x^{-2}$.

$$F(x) = \frac{x^{-2+1}}{-1} = \frac{x^{-1}}{-1} + c = \frac{-1}{x} + c$$

Example 5 : Find the anti derivative $F(x)$ of $f(x) = 1$.

$$F(x) = \frac{1x^{0+1}}{1} = x + c$$

Example 6 : Find the anti derivative of $f(x) = \frac{3}{x^2} + 4x + 5$. Call the anti derivative $F(x)$.

$$\begin{aligned} f(x) &= 3x^{-2} + 4x + 5 \\ F(x) &= \frac{3x^{-1}}{-1} + \frac{4x^2}{2} + 5x + C \\ &= -\frac{3}{x} + 2x^2 + 5x + C \end{aligned}$$

Exercises:

1. Find the anti derivative of each of the following functions

   (a) $6x^2 + 8x - 3$
   (b) $10x^4 - 3x^2 + 5$
   (c) $3x^4 - 6x^2 - 7$
   (d) $x + 3$
   (e) $x^3 - x^{-3} + 2x + 1$
   (f) $\frac{8}{x^3} - \frac{1}{x^2} + 3x + 4$
   (g) $4x^2 - \frac{7}{x^4} + 2$
   (h) $x^4 - 2x$
   (i) $63x^5 - 1$
   (j) $\frac{4}{x^3} - \frac{6}{x^2}$

Section 3 Integration

The area under the curve $y = f(x)$ between $x = a$ and $x = b$, where $f(x) \geq 0$ for $a \leq x \leq b$, is given by the formula

$$A = \int_a^b f(x)\,dx$$

This is read as the integral of the function $f(x)$ from a to b (where a is taken to be the smaller number). The integral can be evaluated using

$$\int_a^b f(x)\,dx = F(x)|_a^b = F(b) - F(a)$$

where $F(x)$ is an anti derivative of $f(x)$.

This is called a definite integral because we integrate between two given values $x = a$ and $x = b$ to obtain a single value. An indefinite integral is written as

$$\int f(x)\,dx = F(x)$$

where again $F(x)$ is an anti derivative of $f(x)$.

Example 1 : Calculate $\int 3x^2\,dx$.

$$\begin{aligned} \int 3x^2\,dx &= \frac{3x^{2+1}}{3} + c \\ &= \frac{3x^3}{3} + c \\ &= x^3 + c \end{aligned}$$

We have used the fact that $\int 3x^2\,dx = 3\int x^2\,dx$. In other words, we can 'pull' the 3 through the integral sign because the 3 is independent of the variable that we are integrating with respect to, which is x in this case. In general $\int af(x)\,dx = a\int f(x)\,dx$.

Note: An indefinite integral is the same as calculating the anti derivative.

Example 2 : Calculate $\int_0^1 (x+3)\,dx$.

$$\begin{aligned}\int_0^1 (x+3)\,dx &= \left(\frac{x^{1+1}}{2} + \frac{3x^{0+1}}{1}\right)\Big]_0^1 \\ &= \left(\frac{x^2}{2} + 3x\right)\Big]_0^1 \\ &= (\frac{1^2}{2} + 3\times 1) - (\frac{0^2}{2} + 3\times 0) \\ &= 3\frac{1}{2}\end{aligned}$$

Example 3 : Calculate the area under the curve $f(x) = x^2$ between $x = 3$ and $x = 6$. The area is given by

$$\begin{aligned}A = \int_3^6 f(x)\,dx &= \int_3^6 x^2\,dx \\ &= \frac{x^3}{3}\Big]_3^6 \\ &= F(6) - F(3) \\ &= \frac{6^3}{3} - \frac{3^3}{3} \\ &= 63\end{aligned}$$

Recall that, in example 1 in section 1, we found that the area was between 58 and 77.

Example 4 : Calculate the area under $f(x) = x$ between $x = 5$ and $x = 7$.

$$\begin{aligned}A = \int_5^7 x\,dx &= \frac{x^2}{2}\Big]_5^7 \\ &= \frac{49}{2} - \frac{25}{2} \\ &= 12\end{aligned}$$

See the example in section 1 for comparison.

Example 5 : Calculate the area under $f(x) = x^4 + x^2$ between $x = -1$ and $x = 0$.

$$\begin{aligned} A = \int_{-1}^{0} (x^4 + x^2)\, dx &= \left(\frac{x^5}{5} + \frac{x^3}{3} \right) \Big]_{-1}^{0} \\ &= \left(\frac{0^5}{5} + \frac{0^3}{3} \right) - \left(\frac{(-1)^5}{5} + \frac{(-1)^3}{3} \right) \\ &= 0 - \left(\frac{-1}{5} - \frac{1}{3} \right) \\ &= \frac{8}{15} \end{aligned}$$

Exercises:

1. Calculate the following integrals

(a) $\int_{-2}^{3} x + 7\, dx$

(b) $\int_{1}^{4} x^2 + 6\, dx$

(c) $\int_{3}^{5} x + 2\, dx$

(d) $\int_{0}^{4} x^2 + x - 1\, dx$

(e) $\int_{-1}^{2} 3x + 4\, dx$

(f) $\int_{0}^{2} 6 - 3x^2\, dx$

(g) $\int_{1}^{3} x^3 - 2x\, dx$

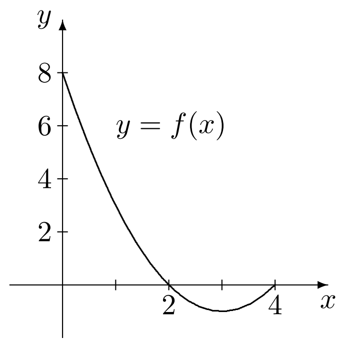
(h) $\int_{0}^{4} x + 2\, dx$

(i) $\int_{-3}^{-1} x^3 + x^2 - 6x\, dx$

(j) $\int_{4}^{6} x + 3\, dx$

Section 4 Integration Continued

As a further investigation of the area under a curve, we will look at the graph of the function $f(x) = x^2 - 6x + 8$.



We will find the area that is shaded. First find the shaded area between $x = 0$ and $x = 2$.

$$\begin{aligned}\int_0^2 x^2 - 6x + 8\,dx &= \left[\frac{x^3}{3} - 3x^2 + 8x\right]_0^2 \\ &= \left(\frac{8}{3} - 12 + 16\right) - (0 - 0 + 0) \\ &= 6\frac{2}{3}\end{aligned}$$

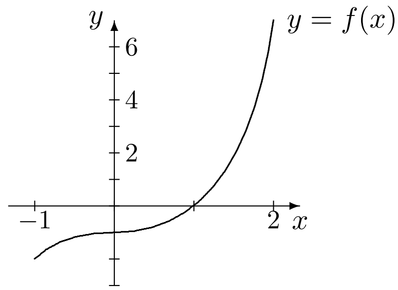
Now see what happens when we use the same method to find the shaded area between $x = 2$ and $x = 4$.

$$\begin{aligned}\int_2^4 x^2 - 6x + 8\,dx &= \left[\frac{x^3}{3} - 3x^2 + 8x\right]_2^4 \\ &= \left(\frac{64}{3} - 48 + 32\right) - \left(\frac{8}{3} - 12 + 16\right) \\ &= -1\frac{1}{3}\end{aligned}$$

An area cannot be negative. The negative sign indicates that the region is below the x axis – in this situation, the actual measure of the area is found by taking the absolute value of the integral. That is, the shaded area between $x = 2$ and $x = 4$ is

$$\left|\int_2^4 x^2 - 6x + 8\,dx\right| = \left|-1\frac{1}{3}\right| = 1\frac{1}{3}$$

Example 1 : Find the area bounded by the curve $y = x^3 - 1$, the x axis, and which lies between the lines $x = 0$ and $x = 1$. First draw the graph.



The required area is below the x axis, so

$$\begin{aligned} A &= \left| \int_0^1 x^3 - 1 \, dx \right| \\ &= \left| \left[\frac{x^4}{4} - x \right]_0^1 \right| \\ &= \left| \left(\frac{1}{4} - 1 \right) - \left(\frac{0}{4} - 0 \right) \right| \\ &= \left| -\frac{3}{4} \right| \\ &= \frac{3}{4} \end{aligned}$$

Example 2 : Find the area bound by the curve $y = x^3 - 1$, the x axis, and the lines $x = 0$ and $x = 3$.

Using the graph from the previous example as a guide, we see that the region from $x = 0$ to $x = 1$ is below the axis, and the region from $x = 1$ to $x = 3$ is above the x axis. So the area we want is

$$\begin{aligned} A &= \left| \int_0^1 x^3 - 1 \, dx \right| + \int_1^3 x^3 - 1 \, dx \\ &= \left| \left[\frac{x^4}{4} - x \right]_0^1 \right| + \left[\frac{x^4}{4} - x \right]_1^3 \\ &= \left| \left(\frac{1}{4} - 1 \right) - \left(\frac{0}{4} - 0 \right) \right| + \left(\frac{81}{4} - 3 \right) - \left(\frac{1}{4} - 1 \right) \\ &= \left| -\frac{3}{4} \right| + 18 \\ &= 18\frac{3}{4} \end{aligned}$$

Exercises for Worksheet 4.2

1. (a) Use the method of left and right rectangles to find upper and lower bounds for the following functions and integration limits:
   i. $y = \sqrt{x}$ between $x = 0$ and $x = 1$ using 5 subdivisions.
   ii. $y = \frac{1}{x}$ between $x = 1$ and $x = 2$ using 10 subdivisions.

   (b) Find the anti derivative of the following functions:

   i. $f(x) = 1 + x + x^2$ ii. $g(x) = x^{\frac{1}{2}}$ iii. $h(x) = \frac{4}{x^3}$

   (c) Evaluate the following definite integrals:

   i. $\displaystyle\int_0^4 7x\,dx$ ii. $\displaystyle\int_0^1 (1 - y^2)\,dy$ iii. $\displaystyle\int_1^2 3t^2\,dt$

2. (a) By using rectangles of width 1, find the area under $y = [x]$ between $x = 0$ and $x = 5$ where $[x]$ is the 'greatest integer' function e.g. $[3.9] = 3$, $[4.1] = 4$.

   (b) Is the function in (i) monotonically increasing?

   (c) Which is greater, $\int_1^2 x\,dx$ or $\int_1^2 \sqrt{x}\,dx$?

   (d) Calculate the area of the region bounded by the graph of $f(x) = (x-2)^2$, the x-axis, and between $x = 2$ and $x = 3$.

   (e) Calculate the area bounded by the curve $y = x^2(3 - x)$ and the x-axis.

   (f) If $\displaystyle\int_{-1}^{a} x\,dx = 0$, evaluate a.

   (g) If $c\displaystyle\int_{-2}^{2} (x - 5)\,dx = 1$, evaluate c.

3. (a) Calculate the area bound by the curves $f(x) = \frac{x^2}{4} - 2$ and $g(x) = x + 1$. (Hint: Find the points of intersection of the two curves, and calculate both areas.)

   (b) Show, by integration, that the area of a unit square is:

   (a) Bisected by the line $y = x$.
   (b) Trisected by the curves $y = x^2$ and $y = \sqrt{x}$.

   (c) The marginal revenue, MR, that a manufacturer receives for his goods is given by $MR = \frac{dR}{dq} = 100 - 0.03q$. Find the total revenue function $R(q)$.

   (d) The density curve of a 10-metre beam is given by $\rho(x) = 3x + 2x^2 - x^{\frac{3}{2}}$ where x is the distance measured from one edge of the beam. The mass of the beam is calculated to be the area under the curve $\rho(x)$ between 0 and x. Find the mass of the beam.

Answers for Worksheet 4.2

Section 1

1. (a) $\frac{2480}{27}; \frac{1328}{27}$ (b) $\frac{161}{2}; \frac{113}{2}$ (c) $\frac{25}{4}; \frac{17}{4}$

Section 2

1. (a) $2x^3 + 4x^2 - 3x + C$
   (b) $2x^5 - x^3 + 5x + C$
   (c) $\frac{3x^5}{5} - 2x^3 - 7x + C$
   (d) $\frac{x^2}{2} + 3x + C$
   (e) $\frac{x^4}{4} + \frac{1}{2x^2} + x^2 + x + C$
   (f) $-\frac{4}{x^2} + \frac{1}{x} + \frac{3x^2}{2} + 4x + C$
   (g) $\frac{4x^3}{3} + \frac{7}{3x^3} + 2x + C$
   (h) $\frac{x^5}{5} - x^2 + C$
   (i) $\frac{63x^6}{6} - x + C$
   (j) $-\frac{2}{x^2} + \frac{6}{x} + C$

Section 3

1. (a) 75/2 (b) 39 (c) 12 (d) 76/3 (e) 33/2 (f) 4 (g) 12 (h) 16 (i) 38/3 (j) 16

Exercises 4.2

1. (a) i. The upper limit is $0.2 \times \left(\sqrt{0.2} + \sqrt{0.4} + \sqrt{0.6} + \sqrt{0.8} + 1\right)$.
   The lower limit is $0.2 \times \left(\sqrt{0.2} + \sqrt{0.4} + \sqrt{0.6} + \sqrt{0.8}\right)$.
   ii. The upper limit is $\frac{1}{10}\sum_{k=0}^{9}\frac{1}{1+\frac{k}{10}}$. The lower limit is $\frac{1}{10}\sum_{k=1}^{10}\frac{1}{1+\frac{k}{10}}$.
   (b) i. $x + \frac{x^2}{2} + \frac{x^3}{3} + C$ ii. $\frac{2x^{3/2}}{3} + C$ iii. $-\frac{2}{x^2} + C$
   (c)

i. 56 ii. 2/3 iii. 7

2. (a) 10
   (b) No
   (c) $\int_1^2 x\,dx$
   (d) 1/3
   (e) 27/4
   (f) 1
   (g) -1/20

3. (a) $-10/3$
   (c) $R = 100q - 0.015q^2 + C$
   (d) $\frac{3x^2}{2} + \frac{2x^3}{3} - \frac{2x^{5/2}}{5}$