Worksheet 3.10 Differentiating Special Functions

Section 1 Exponentials

So far in the worksheets, we have only really covered the differentiation of polynomials. We will need to be able to differentiate other functions as well. This worksheet deals with the rules for differentiating some special functions. How these rules come about will not be shown, as this is a bit complicated for first-year maths.

Recall the properties of the exponential and logarithmic functions:

$$\text{if } \; y = e^x \quad \text{then} \quad x = \log_e y$$

We need to know the derivative of both these functions, which are given by

$$\begin{aligned} \frac{d\,e^x}{dx} &= e^x \\ \text{and} \quad \frac{d\,\log_e x}{dx} &= \frac{1}{x} \end{aligned}$$

These are rules that we need to remember - don't worry about where they come from. They can each be slightly generalized to:

$$\begin{aligned} \frac{d\,e^{g(x)}}{dx} &= g'(x)e^{g(x)} \\ \text{and} \quad \frac{d\,\log_e k(x)}{dx} &= \frac{k'(x)}{k(x)} \end{aligned}$$

Example 1 : Find the derivative of e^{5x^2}. We let $g(x) = 5x^2$. Then $g'(x) = 10x$, and

$$\frac{d\,e^{5x^2}}{dx} = 10xe^{5x^2}$$

Notice that the function which acts as an index - in this case $5x^2$ - is not changed but the entire function e^{5x^2} is multiplied by the derivative of the index.

Example 2 : If $f(x) = e^{4x}$, find $f'(x)$.
$f'(x) = 4e^{4x}$.

Example 3 : Find the derivative of the function $y = \log(6x+3)$. Let $p(x) = 6x+3$. Then $p'(x) = 6$ so that

$$\frac{d\,\log(6x+3)}{dx} = \frac{6}{6x+3}$$

Example 4 : Find the derivative of $f(x) = e^{3x}$.

$$f'(x) = 3e^{3x}$$

Example 5 : Find the derivative of $g(x) = \log(2x^3+3)$.

$$g'(x) = \frac{6x^2}{2x^3+3}$$

Exercises:

1. Differentiate the following with respect to x.

(a) e^{5x}

(b) e^{-2x}

(c) e^{4x^2}

(d) e^{6-x}

(e) e^{4-2x^3}

(f) $\log(4x+1)$

(g) $\log(3x^2-2)$

(h) $\log x^3$

(i) $\log 3x - 4$

(j) $\log e^{4x}$

Section 2 TRIGONOMETRIC FUNCTIONS

The other special functions that you need to know how to differentiate are the trig functions. The rules are:

$$\frac{d\,\sin x}{dx} = \cos x$$

$$\frac{d\,\cos x}{dx} = -\sin x$$

$$\frac{d\,\tan x}{dx} = \sec^2 x$$

These can be generalized to

$$\begin{aligned}\frac{d\,\sin k(x)}{dx} &= k'(x)\cos k(x)\\ \frac{d\,\cos l(x)}{dx} &= -l'(x)\sin l(x)\\ \frac{d\,\tan m(x)}{dx} &= m'(x)\sec^2 m(x)\end{aligned}$$

We will show you how to derive these in the next worksheet. The main ones to remember are those for $\sin x, \cos x$, and $\tan x$.

Example 1 : Find the derivative of $f(x) = \sin 3x$. Then $f'(x) = 3\cos 3x$.

Example 2 : If $g(x) = \cos(x^2)$, then $g'(x) = -2x\sin(x^2)$.

Example 3 : If $h(x) = \tan(3x+2)$, then $h'(x) = 3\sec^2(3x+2)$.

Note: The functions $\sin^2 x$ and $\sin x^2$ are different functions. The notation $\sin^2 x$ means $(\sin x)^2$, and $\sin x^2$ means $\sin(x^2)$. They are both composite functions but they are not equal. Note also that $\sin^2 x$ does not mean $\sin(\sin x)$. This notation holds for the other trig functions as well.

Exercises:

1. Differentiate the following with respect to x.

   (a) $\sin x$
   (b) $\sin 4x$
   (c) $\sin 3x^2$
   (d) $\cos 5x$
   (e) $\cos x^2$
   (f) $\tan x$
   (g) $\tan 4x$
   (h) $\tan(6x+1)$
   (i) $\cos(-3x)$
   (j) $\sin(-6x)$

Exercises 3.10 Differentiating Special Functions

1. Differentiate each of the following functions:

   (a) $f(x) = \log|x|$

   (b) $f(x) = \log(2x - 3)$

   (c) $f(x) = \log|x^2 - 3x + 2|$

   (d) $f(x) = 2e^{x^2}$

   (e) $f(x) = -\frac{1}{3}e^{x^2+2x-1}$

   (f) $f(x) = \sin 3x$

   (g) $f(x) = 3\cos\frac{x}{2}$

   (h) $f(x) = \frac{1}{2}\tan x^2$

   (i) $f(x) = \log x^3 - 2\sin 3x$

   (j) $f(x) = -\frac{1}{2}\cos 2x + \log|3x^2 - 1|$

2. (a) Absolute-value signs often enclose logarithmic expressions. But why? Using your calculator, complete the table below:

x	3.0	2.0	1.0	0.1	0.0	-0.1	-1.0	-2.0
$\ln x$								
$\ln\|x\|$								

   (b) Plot $\ln x$ and $\ln|x|$.

   (c) Why might $y = \ln|x|$ be a better representation of $\int \frac{1}{x}\,dx$ than $y = \ln x$?

3. (a) i. Find the slope of the function $f(x) = 1 - e^x$ at the point where it crosses the x-axis.

   ii. Find the equation of the tangent line to the curve at this point.

   iii. Find the equation of the normal at this point.

   (b) The world's population in 1975 was estimated at 4.1 billion. If the rate of population growth is 0.02, then the population $P(t)$ in billions is given by $P = 4.1e^{0.02t}$, where t is in years.

   i. Calculate the number of years it will take for the population to double.

   ii. Find $\frac{dP}{dt}$, $\left.\frac{dP}{dt}\right]_{t=0}$, and $\left.\frac{dP}{dt}\right]_{t=15}$. What do these quantities represent?

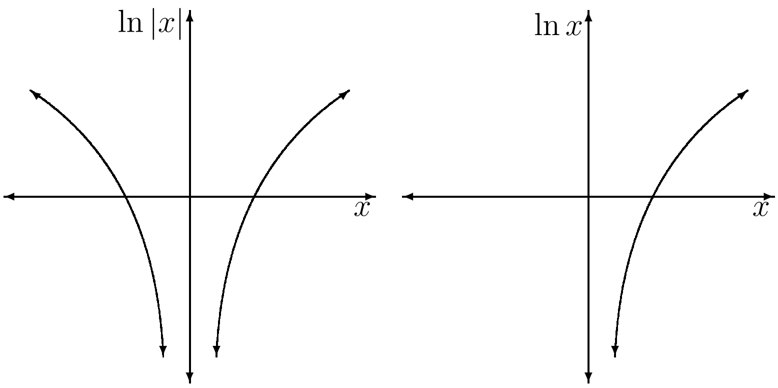
Answers 3.10

1. (a) $\frac{1}{x}$
   (b) $\frac{2}{2x-3}$
   (c) $\frac{2x-3}{x^2-3x+2}$
   (d) $4xe^{x^2}$
   (e) $-\frac{1}{3}(2x+2)e^{x^2+2x-1}$
   (f) $3\cos 3x$
   (g) $-\frac{3}{2}\sin\frac{x}{2}$
   (h) $x\sec^2 x^2$
   (i) $\frac{3}{x} - 6\cos 3x$
   (j) $\sin 2x + \frac{6x}{3x^2-1}$

2. (a)

x	3.0	2.0	1.0	0.1	0.0	-0.1	-1.0	-2.0
$\ln x$	1.098	0.69	0	-2.3	-	-	-	-
$\ln\|x\|$	1.098	0.69	0	-2.3	-	-2.3	0	0.69

   (b)



   (c) $g(x) = \ln|x|$ is defined on the interval $(-\infty, 0)$ and $(0, \infty)$, while $h(x) = \ln x$ is defined on $(0, \infty)$ only. Since $f(x) = \frac{1}{x}$ is defined on the same domain as $g(x)$, then $g(x)$ is a better representation of the integral than $h(x)$.

3. (a) i. -1
   ii. $y = -x$
   iii. $y = x$

   (b) i. $\frac{\ln 2}{0.02} \approx 35$
   ii. $\frac{dP}{dt} = 0.082e^{0.02t}$. This represents the rate of change of population per year (in billions of people per year).
   $\left.\frac{dP}{dt}\right]_{t=0} = 0.082$
   In 1975 the population was increasing at a rate of 82 million people per year.
   $\left.\frac{dP}{dt}\right]_{t=15} = 0.1107$
   In 1990 the population was increasing at a rate of 110.7 million people per year.