Domain and Asymptotes of Rational Functions

A **rational function** is a function that is a fraction of the form $f(x) = \frac{p(x)}{q(x)}$ where p(x) and q(x) are polynomials and q(x) does not equal zero. Some examples of rational functions are as follows:

$$f(x) = \frac{x-1}{x^2-x-2} \qquad g(x) = \frac{1}{x+2} \qquad h(x) = \frac{x}{x^2+1}$$

A. Finding Domain

In general, the domain of a rational function of x includes **<u>all real numbers except x-values that make the denominator equal to zero</u>**. To determine the domain:

1. Set the denominator equal to zero.
2. Solve for x. This may involve a variety of solving methods.
3. These values are **<u>excluded</u>** from the domain.

NOTE: If the function has no variable in the denominator or the solutions of the denominator when set equal to zero are not real numbers, then the domain of the function is all real numbers.

Example: $g(x) = \frac{1}{x+2}$

Step 1: Set the denominator equal to zero. $x + 2 = 0$

$x = -2$ *Solve.*

Step 2: The solution is $= -2$. This is the excluded value.
Step 3: The domain is then the set $\{\, x \mid x \neq -2\}$ or in interval notation as $(-\infty, -2) \cup (-2, \infty)$.

Example: $f(x) = \frac{x-1}{x^2-x-2}$

Step 1: Set denominator equal to zero. $x^2 - x - 2 = 0$

$(x+1)(x-2) = 0$ *Factor to solve.*

$x = -1,\ x = 2$

Step 2: The solutions are $x = -1$ and x = 2. These are the excluded values.
Step 3: The domain is then the set $\{\, x \mid x \neq -1\ ,\ x \neq 2\}$ or in interval notation as $(-\infty, -1) \cup (-1, 2) \cup (2, \infty)$.

B. Finding Vertical Asymptotes

Vertical asymptotes of a function are "invisible" vertical lines that the graph of the function is always approaching but never touching. They serve as boundaries of the function's graph. Vertical asymptotes are found much in the same manner as the domain, except they take on the values of the independent variable *excluded* from the function's domain. To locate the vertical asymptote(s):

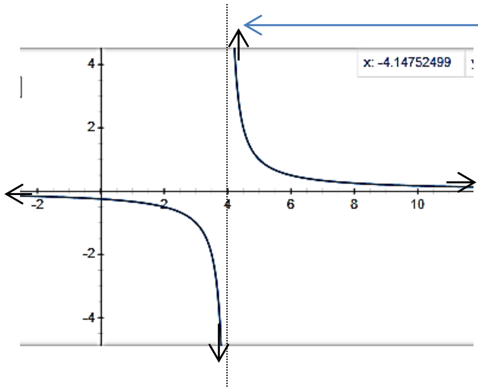
1. Set the denominator equal to zero.
2. Solve for x. This may involve a variety of solving methods.
3. These values are the equations of the vertical lines that are the vertical asymptotes.

Example: $f(x) = \frac{1}{x-4}$

Step 1: Set denominator equal to zero. $x - 4 = 0$

$x = 4$ *Solve.*

Step 2: The solution is $x = 4$.
Step 3: The vertical line $x = 4$ is the vertical asymptote for the graph of the function.



The graph of $f(x) = \frac{1}{x-4}$ approaches the vertical asymptote, $x = 4$, but it does not cross it.

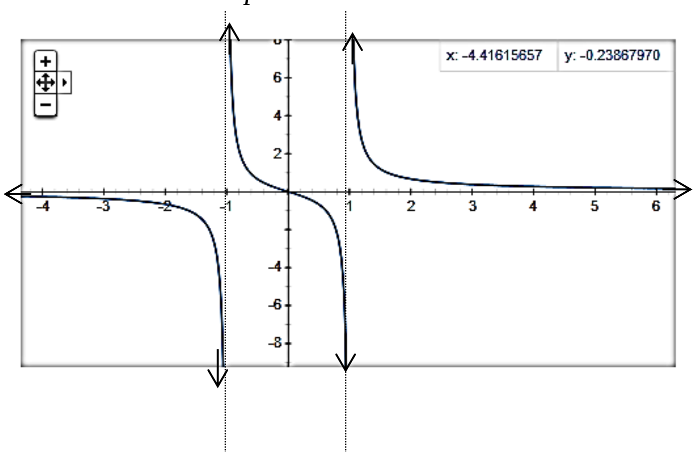
Example: $f(x) = \frac{x}{x^2-1}$

Step 1: Set denominator equal to zero. $x^2 - 1 = 0$

$x = 1, \; x = -1$ *Solve.*

Step 2: The solutions are $x = 1 \; and \; x = -1$.

Step 3: The vertical lines $x = 1 \; and \; x = -1$ are the vertical asymptotes for the graph of the function.



The graph of $f(x) = \frac{x}{x^2-1}$ approaches the vertical asymptotes, $x = 1 \; and \; x = -1$, but it does not cross them.

C. Finding Horizontal Asymptotes

Horizontal asymptotes of a function are "invisible" horizontal lines that the graph of the function is always approaching but never touching at the extreme ends. A graph may cross a horizontal asymptote in the "middle" of the graph. Like vertical asymptotes, they also serve as the boundaries of the function's graph. To determine the horizontal asymptote for a function:

1. Look at the **degree** of polynomials in both the numerator and the denominator. The degree is the highest exponent on the independent variable.
2. There are three possibilities:

a) If the numerator has **lower** degree than the denominator, then the horizontal asymptote is the line $y = 0$ which is the x-axis.

Example: $f(x) = \frac{x-1}{x^2-x-6}$ The degree in the numerator is 1which is less than the denominator which is 2, so y $= 0$ is the horizontal asymptote.

b) If the numerator and denominator have the **same** degree, then the horizontal asymptote is the line $y = \frac{a}{b}$ where a is the leading coefficient in the numerator and b is the leading coefficient in the denominator.

Example: $f(x) = \frac{4x^2-1}{x^2-x-6}$ The degree in the numerator is the same as the denominator, so $y = \frac{4}{1}$ which simplifies to $y = 4$ is the horizontal asymptote.

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c) If the degree of the numerator is **higher** than the degree of the denominator, then the graph of the function has <u>no</u> horizontal asymptote.

Example: $f(x) = \frac{x^4-1}{x^2-x-6}$ The degree of the numerator is greater than the degree of the denominator, so there is NO horizontal asymptote.

D. Finding Slant or Oblique Asymptotes

Slant (also called oblique) asymptotes of a function are straight, diagonal lines that the graph of the function is always approaching but never touching at the extreme ends. They also serve as boundaries of the function's graph. These asymptotes exist *if and only if* the **degree of the numerator is exactly <u>one</u> greater than the degree of the denominator**.

1. Look to see that the degree of the numerator is one greater than the degree of the denominator. If not, there is no slant asymptote.
2. Divide the numerator by the denominator using division of polynomials and drop the remainder. The result is the equation of the line that is the slant asymptote.

Example: $f(x) = \frac{x^3+2x^2+5x-9}{x^2-x+1}$

Step 1: . The degree of the numerator is 3, which is one greater than the degree of the denominator which is 2.

Step 2: When we divide by using long division, the result is seen below. When we drop the remainder, the slant asymptote is the line $= x + 3$.

$$\begin{array}{r} x+3 \\ x^2-x+1 \overline{\big) x^3+2x^2+5x-9} \\ -\left(x^3-x^2+x\right) \\ \hline 3x^2+4x-9 \\ -\left(3x^2-3x+3\right) \\ \hline 7x-12 \end{array}$$