1 Functions

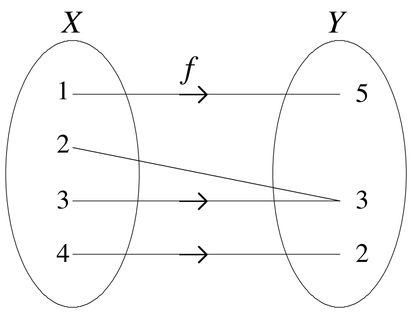
In this Chapter we will cover various aspects of functions. We will look at the definition of a function, the domain and range of a function, what we mean by specifying the domain of a function and absolute value function.

1.1 What is a function?

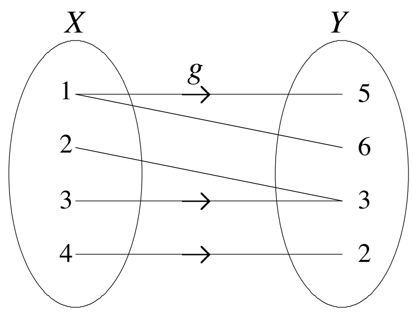
1.1.1 Definition of a function

> A function f from a set of elements X to a set of elements Y is a rule that assigns to each element x in X exactly one element y in Y.

One way to demonstrate the meaning of this definition is by using arrow diagrams.



$f : X \to Y$ is a function. Every element in X has associated with it exactly one element of Y.



$g : X \to Y$ is not a function. The element 1 in set X is assigned two elements, 5 and 6 in set Y.

A function can also be described as a set of ordered pairs (x, y) such that for any x-value in the set, there is only one y-value. This means that there cannot be any repeated x-values with different y-values.

The examples above can be described by the following sets of ordered pairs.

F = {(1,5),(3,3),(2,3),(4,2)} is a function.

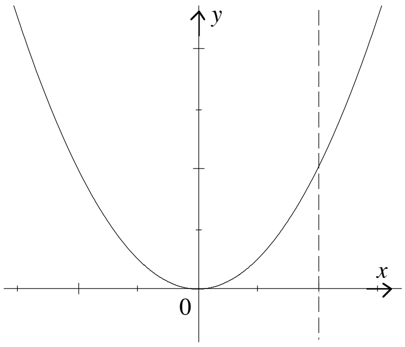
G = {(1,5),(4,2),(2,3),(3,3),(1,6)} is not a function.

The definition we have given is a general one. While in the examples we have used numbers as elements of X and Y, there is no reason why this must be so. However, in these notes we will only consider functions where X and Y are subsets of the real numbers.

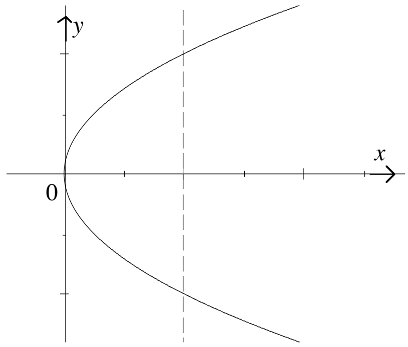
In this setting, we often describe a function using the rule, $y = f(x)$, and create a graph of that function by plotting the ordered pairs $(x, f(x))$ on the Cartesian Plane. This graphical representation allows us to use a test to decide whether or not we have the graph of a function: The Vertical Line Test.

1.1.2 The Vertical Line Test

The Vertical Line Test states that if it is *not possible* to draw a vertical line through a graph so that it cuts the graph in more than one point, then the graph *is* a function.



This is the graph of a function. All possible vertical lines will cut this graph only once.



This is not the graph of a function. The vertical line we have drawn cuts the graph twice.

1.1.3 Domain of a function

For a function $f : X \to Y$ the *domain* of f is the set X.

This also corresponds to the set of x-values when we describe a function as a set of ordered pairs (x, y).

If only the rule $y = f(x)$ is given, then the domain is taken to be the set of all real x for which the function is defined. For example, $y = \sqrt{x}$ has domain; all real $x \geq 0$. This is sometimes referred to as the *natural* domain of the function.

1.1.4 Range of a function

For a function $f : X \to Y$ the *range* of f is the set of y-values such that $y = f(x)$ for some x in X.

This corresponds to the set of y-values when we describe a function as a set of ordered pairs (x, y). The function $y = \sqrt{x}$ has range; all real $y \geq 0$.

Example

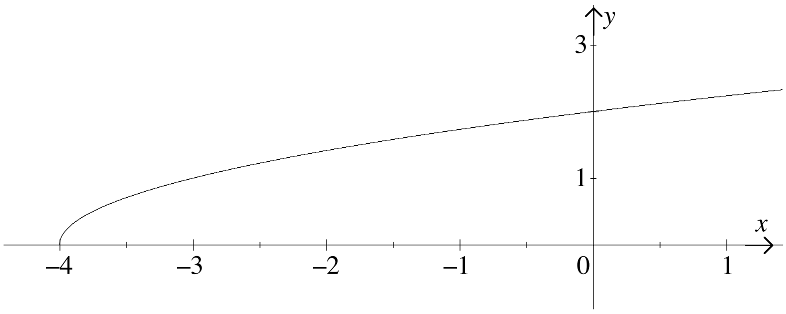
a. State the domain and range of $y = \sqrt{x+4}$.

b. Sketch, showing significant features, the graph of $y = \sqrt{x+4}$.

Solution

a. The domain of $y = \sqrt{x+4}$ is all real $x \geq -4$. We know that square root functions are only defined for positive numbers so we require that $x + 4 \geq 0$, ie $x \geq -4$. We also know that the square root functions are always positive so the range of $y = \sqrt{x+4}$ is all real $y \geq 0$.

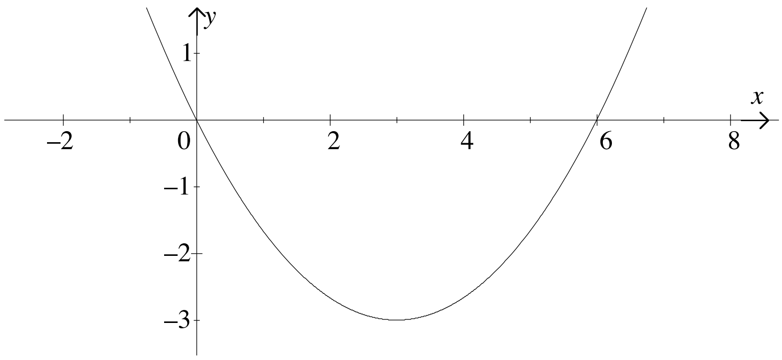
b.



The graph of $y = \sqrt{x+4}$.

Example

a. State the equation of the parabola sketched below, which has vertex $(3, -3)$.



b. Find the domain and range of this function.

Solution

a. The equation of the parabola is $y = \frac{x^2-6x}{3}$.

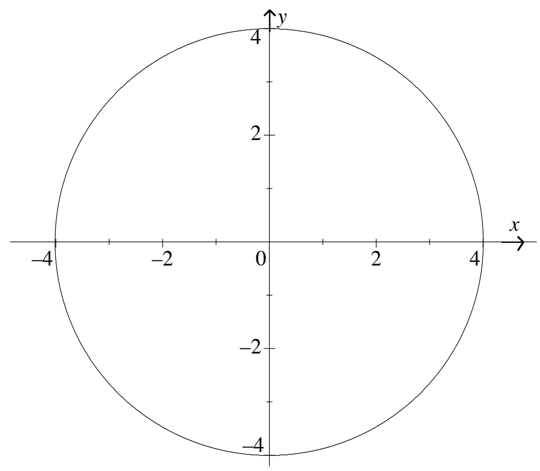
b. The domain of this parabola is all real x. The range is all real $y \geq -3$.

Example

Sketch $x^2 + y^2 = 16$ and explain why it is not the graph of a function.

Solution

$x^2 + y^2 = 16$ is not a function as it fails the vertical line test. For example, when $x = 0$ $y = \pm 4$.



The graph of $x^2 + y^2 = 16$.

Example

Sketch the graph of $f(x) = 3x - x^2$ and find

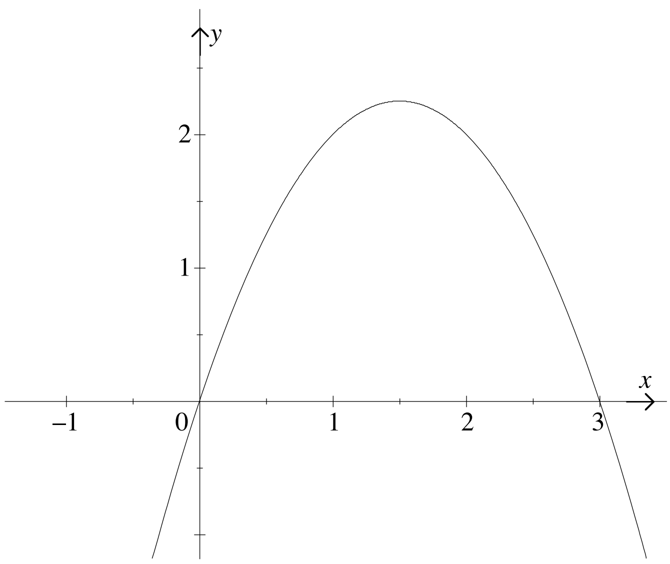
a. the domain and range

b. $f(q)$

c. $f(x^2)$

d. $\frac{f(2+h)-f(2)}{h}$, $h \neq 0$.

Solution



The graph of $f(x) = 3x - x^2$.

a. The domain is all real x. The range is all real y where $y \leq 2.25$.

b. $f(q) = 3q - q^2$

c. $f(x^2) = 3(x^2) - (x^2)^2 = 3x^2 - x^4$

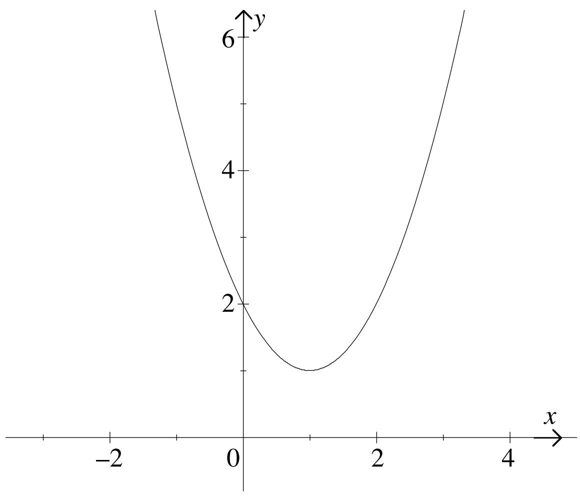
d.

$$\begin{aligned}
\frac{f(2+h)-f(2)}{h} &= \frac{(3(2+h)-(2+h)^2)-(3(2)-(2)^2)}{h} \\
&= \frac{6+3h-(h^2+4h+4)-2}{h} \\
&= \frac{-h^2-h}{h} \\
&= -h-1
\end{aligned}$$

Example

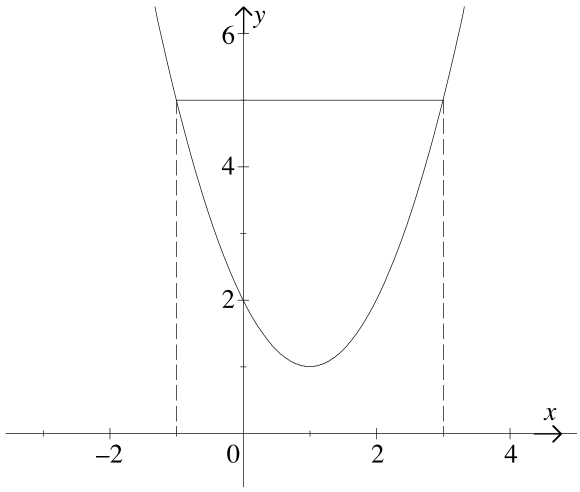
Sketch the graph of the function $f(x) = (x-1)^2 + 1$ and show that $f(p) = f(2-p)$. Illustrate this result on your graph by choosing one value of p.

Solution



The graph of $f(x) = (x-1)^2 + 1$.

$$\begin{aligned}
f(2-p) &= ((2-p)-1)^2 + 1 \\
&= (1-p)^2 + 1 \\
&= (p-1)^2 + 1 \\
&= f(p)
\end{aligned}$$



The sketch illustrates the relationship $f(p) = f(2 - p)$ for $p = -1$. If $p = -1$ then $2 - p = 2 - (-1) = 3$, and $f(-1) = f(3)$.

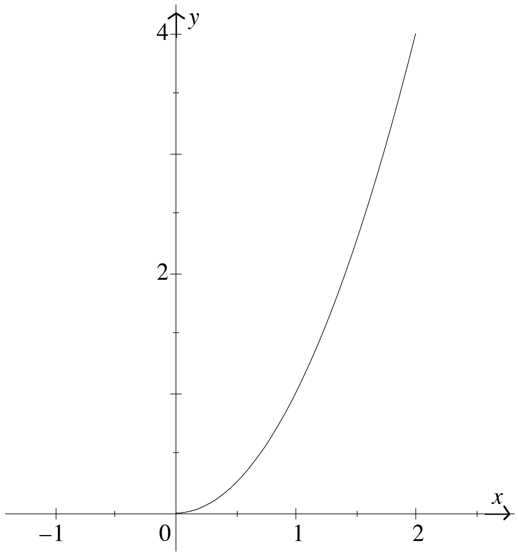
1.2 Specifying or restricting the domain of a function

We sometimes give the rule $y = f(x)$ along with the domain of definition. This domain may not necessarily be the natural domain. For example, if we have the function

$$y = x^2 \quad \text{for} \quad 0 \le x \le 2$$

then the domain is given as $0 \le x \le 2$. The natural domain has been restricted to the subinterval $0 \le x \le 2$.

Consequently, the range of this function is all real y where $0 \le y \le 4$. We can best illustrate this by sketching the graph.



The graph of $y = x^2$ for $0 \le x \le 2$.