12.2 LOGARITHMIC FUNCTIONS AND THEIR APPLICATIONS

In this section

- Definition
- Domain and Range
- Graphing Logarithmic Functions
- Logarithmic Equations
- Applications

In Section 12.1 you learned that exponential functions are one-to-one functions. Because they are one-to-one functions, they have inverse functions. In this section we study the inverses of the exponential functions.

Definition

We define $\log_a(x)$ as *the exponent that is used on the base a to obtain x*. Read $\log_a(x)$ as "the base a logarithm of x." The expression $\log_a(x)$ is called a **logarithm.** Because $2^3 = 8$, the exponent is 3 and $\log_2(8) = 3$. Because $5^2 = 25$, the exponent is 2 and $\log_5(25) = 2$. Because $2^{-5} = \frac{1}{32}$, the exponent is -5 and $\log_2\left(\frac{1}{32}\right) = -5$. So the logarithmic equation $y = \log_a(x)$ is equivalent to the exponential equation $a^y = x$.

$\log_a(x)$

For any $a > 0$ and $a \neq 1$,

$$y = \log_a(x) \quad \text{if and only if} \quad a^y = x.$$

EXAMPLE 1 **Using the definition of logarithm**

Write each logarithmic equation as an exponential equation and each exponential equation as a logarithmic equation.

a) $\log_5(125) = 3$ **b)** $6 = \log_{1/4}(x)$

c) $\left(\frac{1}{2}\right)^m = 8$ **d)** $7 = 3^z$

Solution

a) "The base-5 logarithm of 125 equals 3" means that 3 is the exponent on 5 that produces 125. So $5^3 = 125$.

b) The equation $6 = \log_{1/4}(x)$ is equivalent to $\left(\frac{1}{4}\right)^6 = x$ by the definition of logarithm.

c) The equation $\left(\frac{1}{2}\right)^m = 8$ is equivalent to $\log_{1/2}(8) = m$.

d) The equation $7 = 3^z$ is equivalent to $\log_3(7) = z$. ■

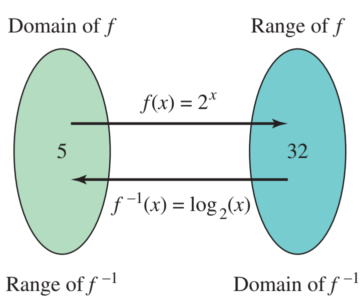


FIGURE 12.8

The inverse of the base-a exponential function $f(x) = a^x$ is the **base-a logarithmic function** $f^{-1}(x) = \log_a(x)$. For example, $f(x) = 2^x$ and $f^{-1}(x) = \log_2(x)$ are inverse functions as shown in Fig. 12.8. Each function undoes the other.

$$f(5) = 2^5 = 32 \quad \text{and} \quad g(32) = \log_2(32) = 5.$$

To evaluate logarithmic functions remember that a logarithm is an exponent: $\log_a(x)$ is the exponent that is used on the base a to obtain x.

EXAMPLE 2 **Finding logarithms**

Evaluate each logarithm.

a) $\log_5(25)$ **b)** $\log_2\left(\frac{1}{8}\right)$ **c)** $\log_{1/2}(4)$

d) $\log_{10}(0.001)$ **e)** $\log_9(3)$

helpful hint

When we write $C(x) = 12x$, we may think of C as a variable and write $C = 12x$, or we may think of C as the name of a function, the cost function. In $y = \log_a(x)$ we are thinking of $\log_a$ only as the name of the function that pairs an x-value with a y-value.

Solution

a) The number $\log_5(25)$ is the exponent that is used on the base 5 to obtain 25. Because $25 = 5^2$, we have $\log_5(25) = 2$.

b) The number $\log_2\left(\frac{1}{8}\right)$ is the power of 2 that gives us $\frac{1}{8}$. Because $\frac{1}{8} = 2^{-3}$, we have $\log_2\left(\frac{1}{8}\right) = -3$.

c) The number $\log_{1/2}(4)$ is the power of $\frac{1}{2}$ that produces 4. Because $4 = \left(\frac{1}{2}\right)^{-2}$, we have $\log_{1/2}(4) = -2$.

d) Because $0.001 = 10^{-3}$, we have $\log_{10}(0.001) = -3$.

e) Because $9^{1/2} = 3$, we have $\log_9(3) = \frac{1}{2}$. ■

There are two bases for logarithms that are used more frequently than the others: They are 10 and e. The base-10 logarithm is called the **common logarithm** and is usually written as $\log(x)$. The base-e logarithm is called the **natural logarithm** and is usually written as $\ln(x)$. Most scientific calculators have function keys for $\log(x)$ and $\ln(x)$. The simplest way to obtain a common or natural logarithm is to use a scientific calculator. However, a table of common logarithms can be found in Appendix C of this text.

In the next example we find natural and common logarithms of certain numbers without a calculator or a table.

EXAMPLE 3 **Finding common and natural logarithms**

Evaluate each logarithm.

a) $\log(1000)$

b) $\ln(e)$

c) $\log\left(\frac{1}{10}\right)$

Solution

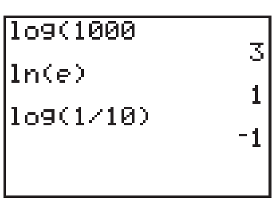
a) Because $10^3 = 1000$, we have $\log(1000) = 3$.

b) Because $e^1 = e$, we have $\ln(e) = 1$.

c) Because $10^{-1} = \frac{1}{10}$, we have $\log\left(\frac{1}{10}\right) = -1$. ■

calculator close-up

A graphing calculator has keys for the common logarithm (LOG) and the natural logarithm (LN).



Domain and Range

The domain of the exponential function $y = 2^x$ is $(-\infty, \infty)$, and its range is $(0, \infty)$. Because the logarithmic function $y = \log_2(x)$ is the inverse of $y = 2^x$, the domain of $y = \log_2(x)$ is $(0, \infty)$, and its range is $(-\infty, \infty)$.

CAUTION Because the domain of $y = \log_a(x)$ is $(0, \infty)$ for any $a > 0$ and $a \neq 1$, expressions such as $\log_2(-4)$, $\log_{1/3}(0)$, and $\ln(-1)$ are undefined.

Graphing Logarithmic Functions

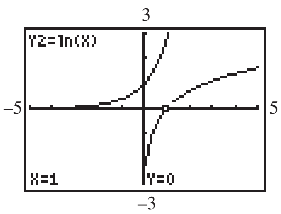
In Chapter 11 we saw that the graphs of a function and its inverse function are symmetric about the line $y = x$. Because the logarithm functions are inverses of exponential functions, their graphs are also symmetric about $y = x$.

EXAMPLE 4 **A logarithmic function with base greater than 1**

Sketch the graph of $g(x) = \log_2(x)$ and compare it to the graph of $y = 2^x$.

calculator close-up

The graphs of $y = \ln(x)$ and $y = e^x$ are symmetric with respect to the line $y = x$. Logarithmic functions with bases other than e and 10 will be graphed on a calculator in Section 12.4.



Solution

Make a table of ordered pairs for $g(x) = \log_2(x)$ using positive numbers for x:

x	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
$g(x) = \log_2(x)$	-2	-1	0	1	2	3

Draw a curve through these points as shown in Fig. 12.9. The graph of the inverse function $y = 2^x$ is also shown in Fig. 12.9 for comparison. Note the symmetry of the two curves about the line $y = x$.

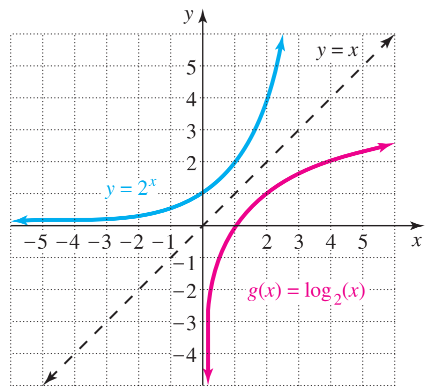


FIGURE 12.9

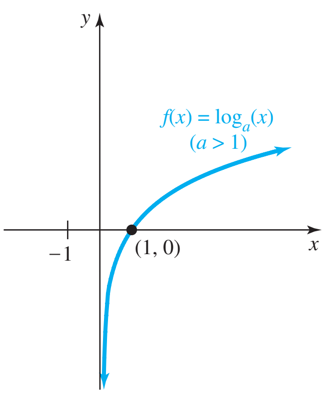


FIGURE 12.10

All logarithmic functions with the base greater than 1 have graphs that are similar to the one in Fig. 12.9. In general, the graph of $f(x) = \log_a(x)$ for $a > 1$ has the following characteristics (see Fig. 12.10):

1. The x-intercept of the curve is (1, 0).
2. The domain is $(0, \infty)$, and the range is $(-\infty, \infty)$.
3. The curve approaches the negative y-axis but does not touch it.
4. The y-values are increasing as we go from left to right along the curve.

EXAMPLE 5

A logarithmic function with base less than 1

Sketch the graph of $f(x) = \log_{1/2}(x)$ and compare it to the graph of $y = \left(\frac{1}{2}\right)^x$.

Solution

Make a table of ordered pairs for $f(x) = \log_{1/2}(x)$ using positive numbers for x:

x	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
$f(x) = \log_{1/2}(x)$	2	1	0	-1	-2	-3

The curve through these points is shown in Fig. 12.11. The graph of the inverse function $y = \left(\frac{1}{2}\right)^x$ is also shown in Fig. 12.11 for comparison. Note the symmetry with respect to the line $y = x$.

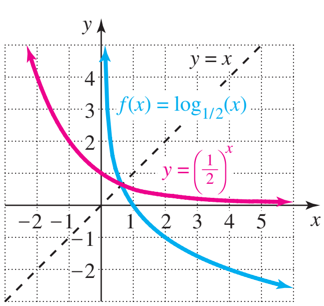


FIGURE 12.11

All logarithmic functions with the base between 0 and 1 have graphs that are similar to the one in Fig. 12.11. In general, the graph of $f(x) = \log_a(x)$ for $0 < a < 1$

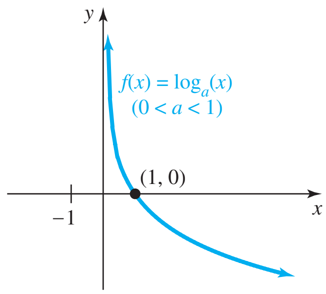


FIGURE 12.12

has the following characteristics (see Fig. 12.12):

1. The x-intercept of the curve is $(1, 0)$.
2. The domain is $(0, \infty)$, and the range is $(-\infty, \infty)$.
3. The curve approaches the positive y-axis but does not touch it.
4. The y-values are decreasing as we go from left to right along the curve.

Figures 12.9 and 12.11 illustrate the fact that $y = \log_a(x)$ and $y = a^x$ are inverse functions for any base a. For any given exponential or logarithmic function the inverse function can be easily obtained from the definition of logarithm.

EXAMPLE 6 **Inverses of logarithmic and exponential functions**

Find the inverse of each function.

a) $f(x) = 10^x$ **b)** $g(x) = \log_3(x)$

Solution

a) To find any inverse function we switch the roles of x and y. So $y = 10^x$ becomes $x = 10^y$. Now $x = 10^y$ is equivalent to $y = \log_{10}(x)$. So the inverse of $f(x) = 10^x$ is $y = \log(x)$ or $f^{-1}(x) = \log(x)$.

b) In $g(x) = \log_3(x)$ or $y = \log_3(x)$ we switch x and y to get $x = \log_3(y)$. Now $x = \log_3(y)$ is equivalent to $y = 3^x$. So the inverse of $g(x) = \log_3(x)$ is $y = 3^x$ or $g^{-1}(x) = 3^x$. ■

Logarithmic Equations

In Section 12.1 we learned that the exponential functions are one-to-one functions. Because logarithmic functions are inverses of exponential functions, they are one-to-one functions also. For a base-a logarithmic function *one-to-one means that if the base-a logarithms of two numbers are equal, then the numbers are equal.*

One-to-One Property of Logarithms

For $a > 0$ and $a \neq 1$,

$$\text{if} \quad \log_a(m) = \log_a(n), \quad \text{then} \quad m = n.$$

The one-to-one property of logarithms and the definition of logarithms are the two basic tools that we use to solve equations involving logarithms. We use these tools in the next example.

EXAMPLE 7 **Logarithmic equations**

Solve each equation.

a) $\log_3(x) = -2$ **b)** $\log_x(8) = -3$ **c)** $\log(x^2) = \log(4)$

Solution

a) Use the definition of logarithms to rewrite the logarithmic equation as an equivalent exponential equation:

$$\log_3(x) = -2$$
$$3^{-2} = x \quad \text{Definition of logarithm}$$
$$\frac{1}{9} = x$$

Because $3^{-2} = \frac{1}{9}$ or $\log_3\left(\frac{1}{9}\right) = -2$, the solution set is $\left\{\frac{1}{9}\right\}$.

study tip

Establish a regular routine of eating, sleeping, and exercise. The ability to concentrate depends on adequate sleep, decent nutrition, and the physical well-being that comes with exercise.

b) Use the definition of logarithms to rewrite the logarithmic equation as an equivalent exponential equation:

$$\begin{aligned}
\log_x(8) &= -3 \\
x^{-3} &= 8 && \text{Definition of logarithm} \\
(x^{-3})^{-1} &= 8^{-1} && \text{Raise each side to the } -1 \text{ power.} \\
x^3 &= \frac{1}{8} \\
x &= \sqrt[3]{\frac{1}{8}} = \frac{1}{2} && \text{Odd-root property}
\end{aligned}$$

Because $\left(\frac{1}{2}\right)^{-3} = 2^3 = 8$ or $\log_{1/2}(8) = -3$ the solution set is $\left\{\frac{1}{2}\right\}$.

c) To write an equation equivalent to $\log(x^2) = \log(4)$, we use the one-to-one property of logarithms:

$$\begin{aligned}
\log(x^2) &= \log(4) \\
x^2 &= 4 && \text{One-to-one property of logarithms} \\
x &= \pm 2 && \text{Even-root property}
\end{aligned}$$

If $x = \pm 2$, then $x^2 = 4$ and $\log(4) = \log(4)$. The solution set is $\{-2, 2\}$. ■

CAUTION If we have equality of two logarithms with the same base, we use the one-to-one property to eliminate the logarithms. If we have an equation with only one logarithm, such as $\log_a(x) = y$, we use the definition of logarithm to write $a^y = x$ and to eliminate the logarithm.

helpful hint

The rule of 70 is used to find approximately how long it takes money to double. Divide 70 by the interest rate, ignoring the percent symbol. For example, at 7% money doubles in approximately $\frac{70}{7}$ or 10 years. To find the time more exactly, divide $\ln(2) \approx 0.693$ by the interest rate.

Applications

When money earns interest compounded continuously, the formula

$$t = \frac{1}{r}\ln\left(\frac{A}{P}\right)$$

expresses the relationship between the time in years t, the annual interest rate r, the principal P, and the amount A. This formula is used to determine how long it takes for a deposit to grow to a specific amount.

EXAMPLE 8 **Finding the time for a specified growth**

How long does it take \$80 to grow to \$240 at 12% compounded continuously?

Solution

Use $r = 0.12$, $P = \$80$, and $A = \$240$ in the formula, and use a calculator to evaluate the logarithm:

$$\begin{aligned}
t &= \frac{1}{0.12}\ln\left(\frac{240}{80}\right) \\
&= \frac{\ln(3)}{0.12} \\
&\approx 9.155
\end{aligned}$$

It takes approximately 9.155 years, or 9 years and 57 days. ■

WARM-UPS

True or false? Explain your answer.

1. The equation $a^3 = 2$ is equivalent to $\log_a(2) = 3$.
2. If (a, b) satisfies $y = 8^x$, then (a, b) satisfies $y = \log_8(x)$.
3. If $f(x) = a^x$ for $a > 0$ and $a \neq 1$, then $f^{-1}(x) = \log_a(x)$.
4. If $f(x) = \ln(x)$, then $f^{-1}(x) = e^x$.
5. The domain of $f(x) = \log_6(x)$ is $(-\infty, \infty)$.
6. $\log_{25}(5) = 2$
7. $\log(-10) = 1$
8. $\log(0) = 0$
9. $5^{\log_5(125)} = 125$
10. $\log_{1/2}(32) = -5$

12.2 EXERCISES

Reading and Writing *After reading this section, write out the answers to these questions. Use complete sentences.*

1. What is the inverse function for the function $f(x) = 2^x$?
2. What is $\log_a(x)$?
3. What is the difference between the common logarithm and the natural logarithm?
4. What is the domain of $f(x) = \log_a(x)$?
5. What is the one-to-one property of logarithmic functions?
6. What is the relationship between the graphs of $f(x) = a^x$ and $f^{-1}(x) = \log_a(x)$ for $a > 0$ and $a \neq 1$?

Write each exponential equation as a logarithmic equation and each logarithmic equation as an exponential equation. See Example 1.

7. $\log_2(8) = 3$ **8.** $\log_{10}(10) = 1$

9. $10^2 = 100$ **10.** $5^3 = 125$

11. $y = \log_5(x)$ **12.** $m = \log_b(N)$

13. $2^a = b$ **14.** $a^3 = c$

15. $\log_3(x) = 10$ **16.** $\log_c(t) = 4$

17. $e^3 = x$ **18.** $m = e^x$

Evaluate each logarithm. See Examples 2 and 3.

19. $\log_2(4)$ **20.** $\log_2(1)$

21. $\log_2(16)$ **22.** $\log_4(16)$

23. $\log_2(64)$ **24.** $\log_8(64)$

25. $\log_4(64)$ **26.** $\log_{64}(64)$

27. $\log_2\left(\frac{1}{4}\right)$ **28.** $\log_2\left(\frac{1}{8}\right)$

29. $\log(100)$ **30.** $\log(1)$

31. $\log(0.01)$ **32.** $\log(10{,}000)$

33. $\log_{1/3}\left(\frac{1}{3}\right)$ **34.** $\log_{1/3}\left(\frac{1}{9}\right)$

35. $\log_{1/3}(27)$ **36.** $\log_{1/3}(1)$

37. $\log_{25}(5)$ **38.** $\log_{16}(4)$

39. $\ln(e^2)$ **40.** $\ln\left(\frac{1}{e}\right)$

Use a calculator to evaluate each logarithm. Round answers to four decimal places.

41. $\log(5)$ **42.** $\log(0.03)$

43. $\ln(6.238)$ **44.** $\ln(0.23)$

Sketch the graph of each function. See Examples 4 and 5.

45. $f(x) = \log_3(x)$

46. $g(x) = \log_{10}(x)$

47. $y = \log_4(x)$

48. $y = \log_5(x)$

49. $h(x) = \log_{1/4}(x)$

50. $y = \log_{1/3}(x)$

51. $y = \log_{1/5}(x)$

52. $y = \log_{1/6}(x)$

Find the inverse of each function. See Example 6.

53. $f(x) = 6^x$

54. $f(x) = 4^x$

55. $f(x) = \ln(x)$

56. $f(x) = \log(x)$

57. $f(x) = \log_{1/2}(x)$

58. $f(x) = \log_{1/4}(x)$

Solve each equation. See Example 7.

59. $x = \left(\frac{1}{2}\right)^{-2}$

60. $x = 16^{-1/2}$

61. $5 = 25^x$

62. $0.1 = 10^x$

63. $\log(x) = -3$

64. $\log(x) = 5$

65. $\log_x(36) = 2$

66. $\log_x(100) = 2$

67. $\log_x(5) = -1$

68. $\log_x(16) = -2$

69. $\log(x^2) = \log(9)$

70. $\ln(2x - 3) = \ln(x + 1)$

Use a calculator to solve each equation. Round answers to four decimal places.

71. $3 = 10^x$

72. $10^x = 0.03$

73. $10^x = \frac{1}{2}$

74. $75 = 10^x$

75. $e^x = 7.2$

76. $e^{3x} = 0.4$

Solve each problem. See Example 8. Use a calculator as necessary.

77. ***Double your money.*** How long does it take \$5000 to grow to \$10,000 at 12% compounded continuously?

78. ***Half the rate.*** How long does it take \$5000 to grow to \$10,000 at 6% compounded continuously?

79. ***Earning interest.*** How long does it take to earn \$1000 in interest on a deposit of \$6000 at 8% compounded continuously?

80. ***Lottery winnings.*** How long does it take to earn \$1000 interest on a deposit of one million dollars at 9% compounded continuously?

The annual growth rate for an investment that is growing continuously is given by

$$r = \frac{1}{t}\ln\left(\frac{A}{P}\right),$$

where P is the principal and A is the amount after t years.

81. ***Top stock.*** An investment of \$10,000 in Dell Computer stock in 1995 grew to \$231,800 in 1998.

a) Assuming the investment grew continuously, what was the annual growth rate?

b) If Dell continues to grow at the same rate, then what will the $10,000 investment be worth in 2002?

82. ***Chocolate bars.*** An investment of $10,000 in 1980 in Hershey stock was worth $563,000 in 1998. Assuming the investment grew continuously, what was the annual growth rate?

In chemistry the pH *of a solution is defined by*

$$\text{pH} = -\log_{10}[H+],$$

where $H+$ is the hydrogen ion concentration of the solution in moles per liter. Distilled water has a pH *of approximately* 7. *A solution with a* pH *under* 7 *is called an acid, and one with a* pH *over* 7 *is called a base.*

83. ***Tomato juice.*** Tomato juice has a hydrogen ion concentration of $10^{-4.1}$ mole per liter (mol/L). Find the pH of tomato juice.

84. ***Stomach acid.*** The gastric juices in your stomach have a hydrogen ion concentration of 10^{-1} mol/L. Find the pH of your gastric juices.

85. ***Neuse River* pH.** The pH of a water sample is one of the many measurements of water quality done by the U.S. Geological Survey. The hydrogen ion concentration of the water in the Neuse River at New Bern, North Carolina, was 1.58×10^{-7} mol/L on July 9, 1998 (Water Resources for North Carolina, wwwnc.usgs.gov). What was the pH of the water at that time?

86. ***Roanoke River* pH.** On July 9, 1998 the hydrogen ion concentration of the water in the Roanoke River at Janesville, North Carolina, was 1.995×10^{-7} mol/L (Water Resources for North Carolina, wwwnc.usgs.gov). What was the pH of the water at that time?

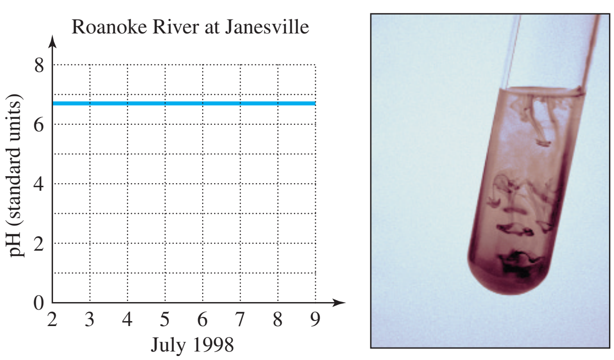


FIGURE FOR EXERCISE 86

Solve each problem.

87. ***Sound level.*** The level of sound in decibels (db) is given by the formula

$$L = 10 \cdot \log(I \times 10^{12}),$$

where I is the intensity of the sound in watts per square meter. If the intensity of the sound at a rock concert is 0.001 watt per square meter at a distance of 75 meters from the stage, then what is the level of the sound at this point in the audience?

88. ***Logistic growth.*** If a rancher has one cow with a contagious disease in a herd of 1000, then the time in days t for n of the cows to become infected is modeled by

$$t = -5 \cdot \ln\left(\frac{1000 - n}{999n}\right).$$

Find the number of days that it takes for the disease to spread to 100, 200, 998, and 999 cows. This model, called a *logistic growth model,* describes how a disease can spread very rapidly at first and then very slowly as nearly all of the population has become infected.

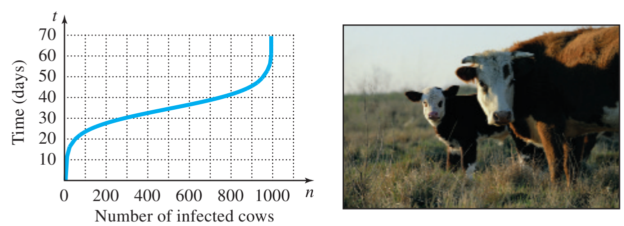


FIGURE FOR EXERCISE 88

GETTING MORE INVOLVED

89. ***Discussion.*** Use the switch-and-solve method from Chapter 11 to find the inverse of the function $f(x) = 5 + \log_2(x - 3)$. State the domain and range of the inverse function.

90. ***Discussion.*** Find the inverse of the function $f(x) = 2 + e^{x+4}$. State the domain and range of the inverse function.

GRAPHING CALCULATOR EXERCISES

91. ***Composition of inverses.*** Graph the functions $y = \ln(e^x)$ and $y = e^{\ln(x)}$. Explain the similarities and differences between the graphs.

92. ***The population bomb.*** The population of the earth is growing continuously with an annual rate of about 1.6%. If the present population is 6 billion, then the function $y = 6e^{0.016x}$ gives the population in billions x years from now. Graph this function for $0 \le x \le 200$. What will the population be in 100 years and in 200 years?