Example 1 Multiplying Monomials

Multiply the monomials.

a. $(3x^2y^7)(5x^3y)$ **b.** $(-3x^4y^3)(-2x^6yz^8)$

Solution:

a. $(3x^2y^7)(5x^3y)$

$= (3 \cdot 5)(x^2 \cdot x^3)(y^7 \cdot y)$ Group coefficients and like bases.

$= 15x^5y^8$ Add exponents and simplify.

b. $(-3x^4y^3)(-2x^6yz^8)$

$= [(-3)(-2)](x^4 \cdot x^6)(y^3 \cdot y)(z^8)$ Group coefficients and like bases.

$= 6x^{10}y^4z^8$ Add exponents and simplify.

Skill Practice Multiply the polynomials.

1. $(-8r^3s)(-4r^4s^4)$ **2.** $(-4ab)(7a^2)$

The distributive property is used to multiply polynomials: $a(b + c) = ab + ac$.

Example 2 Multiplying a Polynomial by a Monomial

Multiply the polynomials.

a. $5y^3(2y^2 - 7y + 6)$ **b.** $-4a^3b^7c\left(2ab^2c^4 - \frac{1}{2}a^5b\right)$

Solution:

a. $5y^3(2y^2 - 7y + 6)$

$= (5y^3)(2y^2) + (5y^3)(-7y) + (5y^3)(6)$ Apply the distributive property.

$= 10y^5 - 35y^4 + 30y^3$ Simplify each term.

b. $-4a^3b^7c\left(2ab^2c^4 - \frac{1}{2}a^5b\right)$

$= (-4a^3b^7c)(2ab^2c^4) + (-4a^3b^7c)\left(-\frac{1}{2}a^5b\right)$ Apply the distributive property.

$= -8a^4b^9c^5 + 2a^8b^8c$ Simplify each term.

Skill Practice Multiply the polynomials.

3. $-6b^2(2b^2 + 3b - 8)$ **4.** $8t^3\left(\frac{1}{2}t^3 - \frac{1}{4}t^2\right)$

Skill Practice Answers

1. $32r^7s^5$ **2.** $-28a^3b$
3. $-12b^4 - 18b^3 + 48b^2$
4. $4t^6 - 2t^5$

Thus far, we have illustrated polynomial multiplication involving monomials. Next, the distributive property will be used to multiply polynomials with more than one term. For example:

$$(x + 3)(x + 5) = (x + 3)x + (x + 3)5$$ Apply the distributive property.

$$= (x + 3)x + (x + 3)5$$ Apply the distributive property again.

$$= x \cdot x + 3 \cdot x + x \cdot 5 + 3 \cdot 5$$

$$= x^2 + 3x + 5x + 15$$

$$= x^2 + 8x + 15$$ Combine *like* terms.

Note: Using the distributive property results in multiplying each term of the first polynomial by each term of the second polynomial:

$$(x + 3)(x + 5) = x \cdot x + x \cdot 5 + 3 \cdot x + 3 \cdot 5$$

$$= x^2 + 5x + 3x + 15$$

$$= x^2 + 8x + 15$$

Example 3 Multiplying Polynomials

Multiply the polynomials.

a. $(2x^2 + 4)(3x^2 - x + 5)$ **b.** $(3y + 2)(7y - 6)$

Solution:

a. $(2x^2 + 4)(3x^2 - x + 5)$ Multiply each term in the first polynomial by each term in the second.

$$= (2x^2)(3x^2) + (2x^2)(-x) + (2x^2)(5) + (4)(3x^2) + (4)(-x) + (4)(5)$$

$$= 6x^4 - 2x^3 + 10x^2 + 12x^2 - 4x + 20$$ Simplify each term.

$$= 6x^4 - 2x^3 + 22x^2 - 4x + 20$$ Combine *like* terms.

TIP: Multiplication of polynomials can be performed vertically by a process similar to column multiplication of real numbers.

$$(2x^2 + 4)(3x^2 - x + 5) \longrightarrow \begin{array}{r} 3x^2 - x + 5 \\ \times\ 2x^2 \quad\quad + 4 \\ \hline 12x^2 - 4x + 20 \\ 6x^4 - 2x^3 + 10x^2 \quad\quad\quad\quad \\ \hline 6x^4 - 2x^3 + 22x^2 - 4x + 20 \end{array}$$

Note: When multiplying by the column method, it is important to align *like* terms vertically before adding terms.

b. $(3y + 2)(7y - 6)$ — Multiply each term in the first polynomial by each term in the second.

$= (3y)(7y) + (3y)(-6) + (2)(7y) + (2)(-6)$ — Apply the distributive property.

$= 21y^2 - 18y + 14y - 12$ — Simplify each term.

$= 21y^2 - 4y - 12$ — Combine *like* terms.

TIP: The acronym, FOIL (**F**irst **O**uter **I**nner **L**ast) can be used as a memory device to multiply two binomials.

Outer terms, First terms, Inner terms, Last terms

First, Outer, Inner, Last

$(3y + 2)(7y - 6) = (3y)(7y) + (3y)(-6) + (2)(7y) + (2)(-6)$

$= 21y^2 - 18y + 14y - 12$

$= 21y^2 - 4y - 12$

Note: It is important to realize that the acronym FOIL may only be used when finding the product of two *binomials*.

Skill Practice Multiply the polynomials.

5. $(5y^2 - 6)(2y^2 - 8y - 1)$ **6.** $(4t + 5)(2t + 3)$

2. Special Case Products: Difference of Squares and Perfect Square Trinomials

In some cases the product of two binomials takes on a special pattern.

I. The first special case occurs when multiplying the sum and difference of the same two terms. For example:

$(2x + 3)(2x - 3)$

$= 4x^2 - 6x + 6x - 9$

$= 4x^2 - 9$

Notice that the "middle terms" are opposites. This leaves only the difference between the square of the first term and the square of the second term. For this reason, the product is called a *difference of squares*.

Definition of Conjugates

The sum and difference of the same two terms are called **conjugates**. For example, we call $2x + 3$ the conjugate of $2x - 3$ and vice versa.

In general, $a + b$ and $a - b$ are conjugates of each other.

Skill Practice Answers

5. $10y^4 - 40y^3 - 17y^2 + 48y + 6$
6. $8t^2 + 22t + 15$

II. The second special case involves the square of a binomial. For example:

$$\left.\begin{aligned}&(3x + 7)^2\\ &= (3x + 7)(3x + 7)\\ &= 9x^2 + 21x + 21x + 49\\ &= 9x^2 \;+\; 42x \;+\; 49\\ &\;\;\uparrow \qquad\;\; \uparrow \qquad\;\; \uparrow\\ &= (3x)^2 + 2(3x)(7) + (7)^2\end{aligned}\right\}$$

When squaring a binomial, the product will be a trinomial called a *perfect square trinomial*. The first and third terms are formed by squaring the terms of the binomial. The middle term is twice the product of the terms in the binomial.

Note: The expression $(3x - 7)^2$ also results in a perfect square trinomial, but the middle term is negative.

$$(3x - 7)(3x - 7) = 9x^2 - 21x - 21x + 49 = 9x^2 - 42x + 49$$

The following table summarizes these special case products.

Special Case Product Formulas

1. $(a + b)(a - b) = a^2 - b^2$ The product is called a **difference of squares**.

2. $(a + b)^2 = a^2 + 2ab + b^2$
$(a - b)^2 = a^2 - 2ab + b^2$ The product is called a **perfect square trinomial**.

It is advantageous for you to become familiar with these special case products because they will be presented again when we factor polynomials.

Example 4 Finding Special Products

Use the special product formulas to multiply the polynomials.

a. $(5x - 2)^2$ **b.** $(6c - 7d)(6c + 7d)$ **c.** $(4x^3 + 3y^2)^2$

Solution:

a. $(5x - 2)^2$ $a = 5x, b = 2$

$= (5x)^2 - 2(5x)(2) + (2)^2$ Apply the formula $a^2 - 2ab + b^2$.

$= 25x^2 - 20x + 4$ Simplify each term.

b. $(6c - 7d)(6c + 7d)$ $a = 6c, b = 7d$

$= (6c)^2 - (7d)^2$ Apply the formula $a^2 - b^2$.

$= 36c^2 - 49d^2$ Simplify each term.

c. $(4x^3 + 3y^2)^2$ $a = 4x^3, b = 3y^2$

$= (4x^3)^2 + 2(4x^3)(3y^2) + (3y^2)^2$ Apply the formula $a^2 + 2ab + b^2$.

$= 16x^6 + 24x^3y^2 + 9y^4$ Simplify each term.

Skill Practice Multiply the polynomials.

7. $(c - 3)^2$ **8.** $(5x - 4)(5x + 4)$ **9.** $(7s^2 + 2t)^2$

The special case products can be used to simplify more complicated algebraic expressions.

Example 5 Using Special Products

Multiply the following expressions.

a. $(x + y)^3$ **b.** $[x + (y + z)][x - (y + z)]$

Solution:

a. $(x + y)^3$

$= (x + y)^2(x + y)$	Rewrite as the square of a binomial and another factor.
$= (x^2 + 2xy + y^2)(x + y)$	Expand $(x + y)^2$ by using the special case product formula.
$= (x^2)(x) + (x^2)(y) + (2xy)(x) + (2xy)(y) + (y^2)(x) + (y^2)(y)$	Apply the distributive property.
$= x^3 + x^2y + 2x^2y + 2xy^2 + xy^2 + y^3$	Simplify each term.
$= x^3 + 3x^2y + 3xy^2 + y^3$	Combine *like* terms.

b. $[x + (y + z)][x - (y + z)]$	This product is in the form $(a + b)(a - b)$, where $a = x$ and $b = (y + z)$.
$= (x)^2 - (y + z)^2$	Apply the formula $a^2 - b^2$.
$= (x)^2 - (y^2 + 2yz + z^2)$	Expand $(y + z)^2$ by using the special case product formula.
$= x^2 - y^2 - 2yz - z^2$	Apply the distributive property.

Skill Practice Multiply the polynomials.

10. $(b + 2)^3$ **11.** $[a + (b + 3)][a - (b + 3)]$

3. Translations Involving Polynomials

Example 6 Translating Between English Form and Algebraic Form

Complete the table.

English Form	Algebraic Form
The square of the sum of x and y	
	$x^2 + y^2$
The square of the product of 3 and x	

Skill Practice Answers

7. $c^2 - 6c + 9$ **8.** $25x^2 - 16$
9. $49s^4 + 28s^2t + 4t^2$
10. $b^3 + 6b^2 + 12b + 8$
11. $a^2 - b^2 - 6b - 9$

Solution:

English Form	Algebraic Form	Notes
The square of the sum of x and y	$(x + y)^2$	The *sum* is squared, not the individual terms.
The sum of the squares of x and y	$x^2 + y^2$	The individual terms x and y are squared first. Then the sum is taken.
The square of the product of 3 and x	$(3x)^2$	The product of 3 and x is taken. Then the result is squared.

Skill Practice Translate to algebraic form:

12. The square of the difference of a and b

13. The difference of the square of a and the square of b

14. Translate to English form: $a - b^2$.

4. Applications Involving a Product of Polynomials

Example 7 Applying a Product of Polynomials

A box is created from a sheet of cardboard 20 in. on a side by cutting a square from each corner and folding up the sides (Figures 5-3 and 5-4). Let x represent the length of the sides of the squares removed from each corner.

a. Find an expression for the volume of the box in terms of x.

b. Find the volume if a 4-in. square is removed.

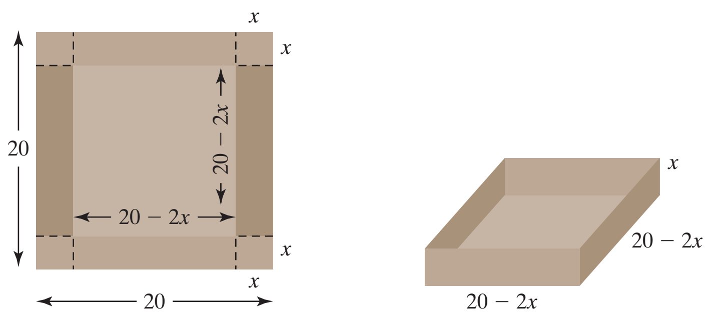


Figure 5-3 **Figure 5-4**

Solution:

a. The volume of a rectangular box is given by the formula $V = lwh$. The length and width can both be expressed as $20 - 2x$. The height of the box is x. Hence the volume is given by

$$\begin{aligned} V &= l \cdot w \cdot h \\ &= (20 - 2x)(20 - 2x)x \\ &= (20 - 2x)^2 x \\ &= (400 - 80x + 4x^2)x \\ &= 400x - 80x^2 + 4x^3 \\ &= 4x^3 - 80x^2 + 400x \end{aligned}$$

Skill Practice Answers

12. $(a - b)^2$ **13.** $a^2 - b^2$

14. The difference of a and the square of b

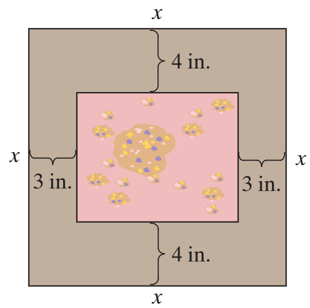
b. If a 4-in. square is removed from the corners of the box, we have $x = 4$ in. The volume is

$$\begin{aligned} V &= 4(4)^3 - 80(4)^2 + 400(4) \\ &= 4(64) - 80(16) + 400(4) \\ &= 256 - 1280 + 1600 \\ &= 576 \end{aligned}$$

The volume is 576 in.^3

Skill Practice

15. A rectangular photograph is mounted on a square piece of cardboard whose sides have length x. The border that surrounds the photo is 3 in. on each side and 4 in. on both top and bottom.



a. Write an expression for the area of the photograph and multiply.
b. Determine the area of the photograph if x is 12.

Skill Practice Answers

15a. $A = (x - 8)(x - 6)$; $A = x^2 - 14x + 48$
b. 24 in.^2