4.4 Difference of Squares and Perfect Square Trinomials

4.4 OBJECTIVES

1. Factor a binomial that is the difference of two squares
2. Factor a perfect square trinomial

In Section 3.5, we introduced some special products. Recall the following formula for the product of a sum and difference of two terms:

$$(a + b)(a - b) = a^2 - b^2 \tag{1}$$

This also means that a binomial of the form $a^2 - b^2$, called a **difference of two squares,** has as its factors $a + b$ and $a - b$.

To use this idea for factoring, we can write

$$a^2 - b^2 = (a + b)(a - b) \tag{2}$$

A **perfect square** term has a coefficient that is a square (1, 4, 9, 16, 25, 36, etc.), and any variables have exponents that are multiples of 2 (x^2, y^4, z^6, etc.).

Identifying Perfect Square Terms

For each of the following, decide whether it is a perfect square term. If it is, find the expression that was squared (called the *root*).

(a) $36x$
(b) $24x^6$
(c) $9x^4$
(d) $64x^6$
(e) $16x^9$

Only parts c and d are perfect square terms.

$9x^4 = (3x^2)^2$

$64x^6 = (8x^3)^2$

CHECK YOURSELF 1

For each of the following, decide whether it is a perfect square term. If it is, find the expression that was squared.

(a) $36x^{12}$ **(b)** $4x^6$
(c) $9x^7$ **(d)** $25x^8$
(e) $16x^{25}$

We will now use equation 2 above to factor the difference between two perfect square terms.

Example 2

Factoring the Difference of Two Squares

Factor $x^2 - 16$.

Think $x^2 - 4^2$

NOTE You could also write $(x - 4)(x + 4)$. The order doesn't matter because multiplication is commutative.

Because $x^2 - 16$ is a difference of squares, we have

$$x^2 - 16 = (x + 4)(x - 4)$$

CHECK YOURSELF 2

Factor $m^2 - 49$.

Any time an expression is a difference of two squares, it can be factored.

Example 3

Factoring the Difference of Two Squares

Factor $4a^2 - 9$.

Think $(2a)^2 - 3^2$

So

$$4a^2 - 9 = (2a)^2 - (3)^2$$
$$= (2a + 3)(2a - 3)$$

CHECK YOURSELF 3

Factor $9b^2 - 25$.

The process for factoring a difference of squares does not change when more than one variable is involved.

Factoring the Difference of Two Squares

NOTE Think $(5a)^2 - (4b^2)^2$

Factor $25a^2 - 16b^4$.

$$25a^2 - 16b^4 = (5a + 4b^2)(5a - 4b^2)$$

CHECK YOURSELF 4

Factor $49c^4 - 9d^2$.

We will now consider an example that combines common-term factoring with difference-of-squares factoring. Note that the common factor is always removed as the *first step*.

Example 5

Removing the GCF First

NOTE Step 1
Remove the GCF.
Step 2
Factor the remaining binomial.

Factor $32x^2y - 18y^3$.

Note that $2y$ is a common factor, so

$$32x^2y - 18y^3 = 2y(\underbrace{16x^2 - 9y^2}_{\text{Difference of squares}})$$

$$= 2y(4x + 3y)(4x - 3y)$$

CHECK YOURSELF 5

Factor $50a^3 - 8ab^2$.

CAUTION

Note that this is different from the sum of two squares (like $x^2 + y^2$), which never has integer factors.

Recall the following multiplication pattern.

$$(a + b)^2 = a^2 + 2ab + b^2$$

For example,

$$(x + 2)^2 = x^2 + 4x + 4$$

$$(x + 5)^2 = x^2 + 10x + 25$$

$$(2x + 1)^2 = 4x^2 + 4x + 1$$

Recognizing this pattern can simplify the process of factoring perfect square trinomials.

Factoring a Perfect Square Trinomial

Factor the trinomial $4x^2 + 12xy + 9y^2$.

Note that this is a perfect square trinomial in which

$a = 2x$ and $b = 3y$.

In factored form, we have

$$4x^2 + 12xy + 9y^2 = (2x + 3y)^2$$

CHECK YOURSELF 6

Factor the trinomial $16u^2 + 24uv + 9v^2$.

Recognizing the same pattern can simplify the process of factoring perfect square trinomials in which the second term is negative.

Factoring a Perfect Square Trinomial

Factor the trinomial $25x^2 - 10xy + y^2$.

This is also a perfect square trinomial, in which

$a = 5x$ and $b = -y$.

In factored form, we have

$$25x^2 - 10xy + y^2 = (5x + (-y))^2 = (5x - y)^2$$

CHECK YOURSELF 7

Factor the trinomial $4u^2 - 12uv + 9v^2$.

CHECK YOURSELF ANSWERS

1. **(a)** $(6x^6)^2$; **(b)** $(2x^3)^2$; **(d)** $(5x^4)^2$ **2.** $(m + 7)(m - 7)$ **3.** $(3b + 5)(3b - 5)$
4. $(7c^2 + 3d)(7c^2 - 3d)$ **5.** $2a(5a + 2b)(5a - 2b)$ **6.** $(4u + 3v)^2$
7. $(2u - 3v)^2$

4.4 Exercises

Name ______________________

Section __________ Date __________

For each of the following binomials, state whether the binomial is a difference of squares.

1. $3x^2 + 2y^2$

2. $5x^2 - 7y^2$

3. $16a^2 - 25b^2$

4. $9n^2 - 16m^2$

5. $16r^2 + 4$

6. $p^2 - 45$

7. $16a^2 - 12b^3$

8. $9a^2b^2 - 16c^2d^2$

9. $a^2b^2 - 25$

10. $4a^3 - b^3$

Factor the following binomials.

11. $m^2 - n^2$

12. $r^2 - 9$

13. $x^2 - 49$

14. $c^2 - d^2$

15. $49 - y^2$

16. $81 - b^2$

17. $9b^2 - 16$

18. $36 - x^2$

19. $16w^2 - 49$

20. $4x^2 - 25$

21. $4s^2 - 9r^2$

22. $64y^2 - x^2$

23. $9w^2 - 49z^2$

24. $25x^2 - 81y^2$

ANSWERS

1. ______
2. ______
3. ______
4. ______
5. ______
6. ______
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ANSWERS

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50. ______
51. ______
52. ______

25. $16a^2 - 49b^2$

26. $64m^2 - 9n^2$

27. $x^4 - 36$

28. $y^6 - 49$

29. $x^2y^2 - 16$

30. $m^2n^2 - 64$

31. $25 - a^2b^2$

32. $49 - w^2z^2$

33. $r^4 - 4s^2$

34. $p^2 - 9q^4$

35. $81a^2 - 100b^6$

36. $64x^4 - 25y^4$

37. $18x^3 - 2xy^2$

38. $50a^2b - 2b^3$

39. $12m^3n - 75mn^3$

40. $63p^4 - 7p^2q^2$

41. $48a^2b^2 - 27b^4$

42. $20w^5 - 45w^3z^4$

Determine whether each of the following trinomials is a perfect square. If it is, factor the trinomial.

43. $x^2 - 14x + 49$

44. $x^2 + 9x + 16$

45. $x^2 - 18x - 81$

46. $x^2 + 10x + 25$

47. $x^2 - 18x + 81$

48. $x^2 - 24x + 48$

Factor the following trinomials.

49. $x^2 + 4x + 4$

50. $x^2 + 6x + 9$

51. $x^2 - 10x + 25$

52. $x^2 - 8x + 16$

53. $4x^2 + 12xy + 9y^2$

54. $16x^2 + 40xy + 25y^2$

55. $9x^2 - 24xy + 16y^2$

56. $9w^2 - 30wv + 25v^2$

57. $y^3 - 10y^2 + 25y$

58. $12b^3 - 12b^2 + 3b$

Factor each expression.

59. $x^2(x + y) - y^2(x + y)$

60. $a^2(b - c) - 16b^2(b - c)$

61. $2m^2(m - 2n) - 18n^2(m - 2n)$

62. $3a^3(2a + b) - 27ab^2(2a + b)$

63. Find a value for k so that $kx^2 - 25$ will have the factors $2x + 5$ and $2x - 5$.

64. Find a value for k so that $9m^2 - kn^2$ will have the factors $3m + 7n$ and $3m - 7n$.

65. Find a value for k so that $2x^3 - kxy^2$ will have the factors $2x$, $x - 3y$, and $x + 3y$.

66. Find a value for k so that $20a^3b - kab^3$ will have the factors $5ab$, $2a - 3b$, and $2a + 3b$.

67. Complete the following statement in complete sentences: "To factor a number you . . . ."

68. Complete this statement: To factor an algebraic expression into prime factors means . . . .

Getting Ready for Section 4.5 [Section 4.1]

Factor.

(a) $2x(3x + 2) - 5(3x + 2)$

(b) $3y(y - 4) + 5(y - 4)$

(c) $3x(x + 2y) + y(x + 2y)$

(d) $5x(2x - y) - 3(2x - y)$

(e) $4x(2x - 5y) - 3y(2x - 5y)$

ANSWERS

53. ____
54. ____
55. ____
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65. ____
66. ____
67. ____
68. ____
a. ____
b. ____
c. ____
d. ____
e. ____

Answers

1. No **3.** Yes **5.** No **7.** No **9.** Yes **11.** $(m + n)(m - n)$
13. $(x + 7)(x - 7)$ **15.** $(7 + y)(7 - y)$ **17.** $(3b + 4)(3b - 4)$
19. $(4w + 7)(4w - 7)$ **21.** $(2s + 3r)(2s - 3r)$ **23.** $(3w + 7z)(3w - 7z)$
25. $(4a + 7b)(4a - 7b)$ **27.** $(x^2 + 6)(x^2 - 6)$ **29.** $(xy + 4)(xy - 4)$
31. $(5 + ab)(5 - ab)$ **33.** $(r^2 + 2s)(r^2 - 2s)$ **35.** $(9a + 10b^3)(9a - 10b^3)$
37. $2x(3x + y)(3x - y)$ **39.** $3mn(2m + 5n)(2m - 5n)$
41. $3b^2(4a + 3b)(4a - 3b)$ **43.** Yes; $(x - 7)^2$ **45.** No **47.** Yes; $(x - 9)^2$
49. $(x + 2)^2$ **51.** $(x - 5)^2$ **53.** $(2x + 3y)^2$ **55.** $(3x - 4y)^2$
57. $y(y - 5)^2$ **59.** $(x + y)^2(x - y)$ **61.** $2(m - 2n)(m + 3n)(m - 3n)$
63. 4 **65.** 18 **67.**  **a.** $(3x + 2)(2x - 5)$

b. $(y - 4)(3y + 5)$ **c.** $(x + 2y)(3x + y)$ **d.** $(2x - y)(5x - 3)$
e. $(2x - 5y)(4x - 3y)$