11.1 Inverse Relations and Functions

11.1 OBJECTIVES

1. Find the inverse of a relation
2. Graph a relation and its inverse
3. Find the inverse of a function
4. Graph a function and its inverse
5. Identify a one-to-one function

Let's consider an extension of the concepts of relations and functions discussed in Chapter 3.

Suppose we are given the relation

$$\{(1, 2), (2, 4), (3, 6)\} \tag{1}$$

If we *interchange* the first and second components (the x and y values) of each of the ordered pairs in relation (1), we have

$$\{(2, 1), (4, 2), (6, 3)\} \tag{2}$$

which is another relation. Relations (1) and (2) are called **inverse relations,** and in general we have the following definition.

Definitions: Inverse of a Relation

The *inverse* of a relation is formed by interchanging the components of each of the ordered pairs in the given relation.

Because we know that relations are often specified by equations, it is natural for us to want to work with the concept of the inverse relation in that setting. We form the inverse relation by interchanging the roles of x and y in the defining equation. Example 1 illustrates this concept.

Example 1

Finding the Inverse of a Relation

Find the inverse of the relation.

$$f = \{(x, y) \mid y = 2x - 4\} \tag{3}$$

First interchange variables x and y to obtain

NOTE Notice that x and y have been interchanged from the original equation.

$$x = 2y - 4$$

We now solve the defining equation for y.

$$2y = x + 4$$

$$y = \frac{1}{2}x + 2$$

Then, we rewrite the relation in the equivalent form.

$$f^{-1} = \left\{(x, y) \,\middle|\, y = \frac{1}{2}x + 2\right\} \tag{4}$$

NOTE The notation f^{-1} has a *different meaning* from the negative exponent, as in x^{-1} or $\dfrac{1}{x}$.

The inverse of the original relation (3) is now shown in (4) with the defining equation "solved for y." That inverse is denoted f^{-1} (this is read as "f inverse").
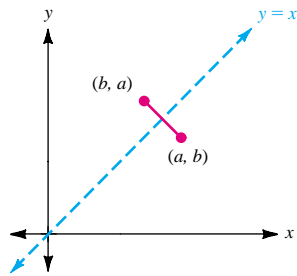
We use the notation f^{-1} to indicate the inverse of f when that inverse is *also a function.*

CHECK YOURSELF 1

Write the inverse relation for $g = \{(x, y) \mid y = 3x + 6\}$.

The graphs of relations and their inverses are related in an interesting way. First, note that the graphs of the ordered pairs (a, b) and (b, a) always have symmetry about the line $y = x$.

Now, with the above symmetry in mind, let's consider Example 2.

Example 2

Graphing a Relation and Its Inverse

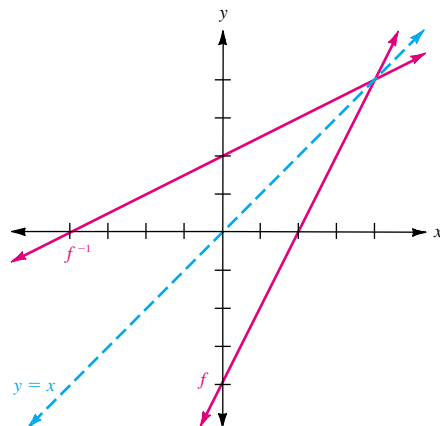Graph the relation f from Example 1 along with its inverse.
Recall that

$$f = \{(x, y) \mid y = 2x - 4\}$$

and

$$f^{-1} = \left\{(x, y) \mid y = \frac{1}{2}x + 2\right\}$$

The graphs of f and f^{-1} are shown below.

Note that the graphs of f and f^{-1} are symmetric about the line $y = x$. That symmetry follows from our earlier observation about the pairs (a, b) and (b, a) because we simply reversed the roles of x and y in forming the inverse relation.

CHECK YOURSELF 2

Graph the relation g from the Check Yourself 1 *exercise along with its inverse.*

From our work thus far, it should be apparent that every relation has an inverse. However, that inverse may or may not be a function.

Example 3

Finding the Inverse of a Function

Find the inverses of the following functions.

(a) $f = \{(1, 3), (2, 4), (3, 9)\}$

Its inverse is

NOTE The elements of the ordered pairs have been interchanged.

$\{(3, 1), (4, 2), (9, 3)\}$

which is also a function.

(b) $g = \{(1, 3), (2, 6), (3, 6)\}$

Its inverse is

NOTE It is not a function because 6 is mapped to both 2 and 3.

$\{(3, 1), (6, 2), (6, 3)\}$

which is *not* a function.

CHECK YOURSELF 3

Write the inverses for each of the following relations. Which of the inverses are also functions?

(a) $\{(-1, 2), (0, 3), (1, 4)\}$ **(b)** $\{(2, 5), (3, 7), (4, 5)\}$

Can we predict in advance whether the inverse of a function will also be a function? The answer is yes.

We already know that for a relation to be a function, no element in its domain can be associated with more than one element in its range.

In addition, if the inverse of a function is to be a function, no element in the range can be associated with more than one element in the domain—that is, no two distinct ordered pairs in the function can have the same second component. A function that satisfies this additional restriction is called a **one-to-one function.**

The function in Example 3(a)

$f = \{(1, 3), (2, 4), (3, 9)\}$

is a one-to-one function and its inverse is also a function. However, the function in Example 3(b)

$g = \{(1, 3), (2, 6), (3, 6)\}$

is *not* a one-to-one function, and its inverse is *not* a function.

From those observations we can state the following general result.

Rules and Properties: Inverse of a Function

A function f has an inverse f^{-1}, which is also a function, if and only if f is a one-to-one function.

Because the statement is an "if and only if" statement, it can be turned around without changing the meaning. Here we use the same statement as a definition for a one-to-one function.

NOTE This tells us that the inverse of f is also a function.

Definitions: One-To-One Function

A function f is a *one-to-one function* if and only if no two distinct domain elements are paired with the same range element.

Our result regarding a one-to-one function and its inverse also has a convenient graphical interpretation, as Example 4 illustrates.

Example 4

Graphing a Function and Its Inverse

Graph each function and its inverse. State which inverses are functions.

(a) $f = \{(x, y)\,|\,y = 4x - 8\}$

Because f is a one-to-one function (no value for y can be associated with more than one value for x), its inverse is also a function. Here,

$$f^{-1} = \left\{(x, y)\,\Big|\,y = \frac{1}{4}x + 2\right\}$$

This is a **linear function** of the form $f = \{(x, y)\,|\,y = mx + b\}$. Its graph is a straight line. A linear function, in which $m \neq 0$, is always one-to-one.

The graphs of f and f^{-1} are shown below.
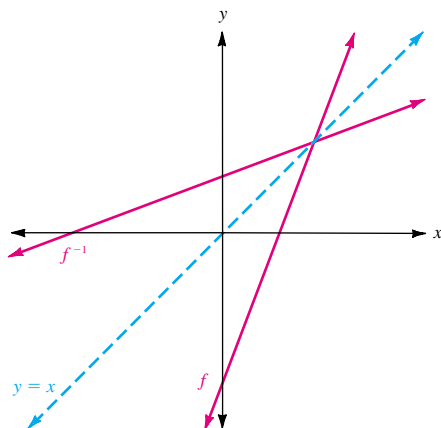
NOTE The vertical-line test tells us that *both f and f^{-1}* are functions.

(b) $g = \{(x, y)\,|\,y = x^2\}$

This is a **quadratic function** of the form

$g = \{(x, y)\,|\,y = ax^2 + bx + c\}$ in which $a \neq 0$

Its graph is always a parabola, and a quadratic function is *not* a one-to-one function.

For instance, 4 in the range is associated with both 2 and -2 from the domain. It follows that the inverse of g
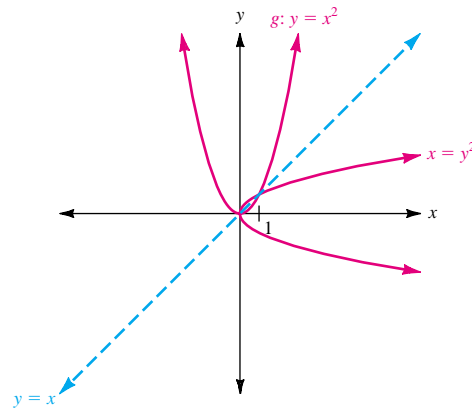
$\{(x, y)\,|\,x = y^2\}$

or

$\{(x, y)\,|\,y = \pm\sqrt{x}\}$

is *not* a function. The graphs of g and its inverse are shown below.

Note: When a function is not one-to-one, as in Example 4(b), we can restrict the domain of the function so that it will be one-to-one. In this case, if we redefine function g as

NOTE The domain is now restricted to nonnegative values for *x*.

$g = \{(x, y)\,|\,y = x^2, x \geq 0\}$

it will be one-to-one and its inverse

$g^{-1} = \{(x, y)\,|\,y = \sqrt{x}\}$

will be a function, as shown in the following graph.
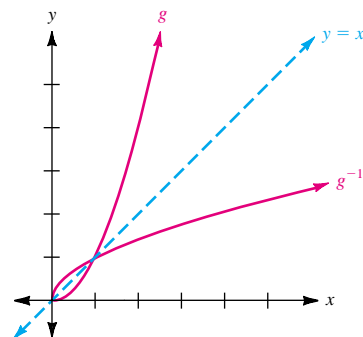
NOTE The function g is now one-to-one, and its inverse g^{-1} is also a function.

CHECK YOURSELF 4

Graph each function and its inverse. Which inverses are functions?

(a) $f = \{(x, y)\,|\,y = 2x - 2\}$ **(b)** $g = \{(x, y)\,|\,y = 2x^2\}$

NOTE In Chapter 3, we referred to the *vertical line test* to determine whether a relation was a function. The *horizontal line test* determines whether a function is one-to-one.
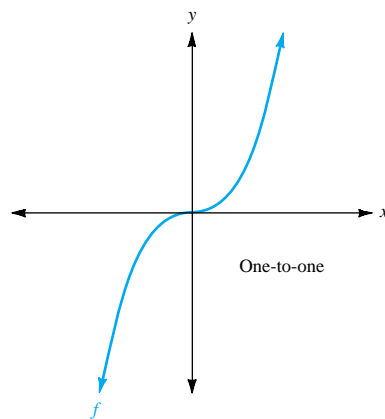
It is easy to tell from the graph of a function whether that function is one-to-one. If any horizontal line can meet the graph of a function in at most one point, the function is one-to-one. Example 5 illustrates this approach.

Example 5

Identifying a One-to-One Function

Which of the following graphs represent one-to-one functions?

(a)

One-to-one

Because no horizontal line passes through any two points of the graph, *f* is one-to-one.

(b)
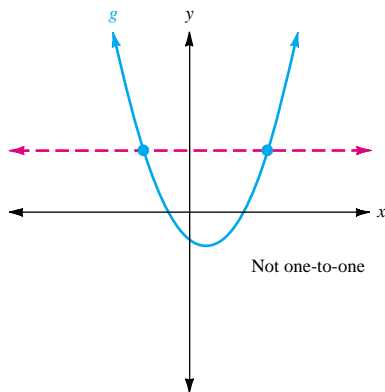
Not one-to-one

Because a horizontal line can meet the graph of function *g* at two points, *g* is *not* a one-to-one function.

CHECK YOURSELF 5

*Consider the graphs of the functions of Check Yourself 4. Which functions are one-to-one?*

The following algorithm summarizes our work in this section.

Step by Step: Finding Inverse Relations and Functions

Step 1   Interchange the x and y components of the ordered pairs of the given relation or the roles of x and y in the defining equation.

Step 2   If the relation was described in equation form, solve the defining equation of the inverse for y.

Step 3   If desired, graph the relation and its inverse on the same set of axes. The two graphs will be symmetric about the line $y = x$.
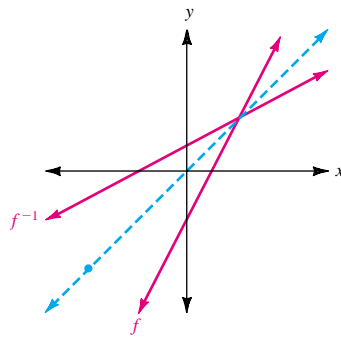
CHECK YOURSELF ANSWERS

1. $g^{-1} = \left\{ (x, y) \mid y = \dfrac{1}{3}x - 2 \right\}$ **2.**
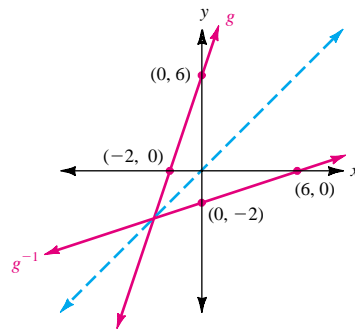
3. (a) $\{(2, -1), (3, 0), (4, 1)\}$—a function; **(b)** $\{(5, 2), (7, 3), (5, 4)\}$—*not* a function

4. (a) $f = \{(x, y) \mid y = 2x - 2\}$, $f^{-1} = \left\{ (x, y) \mid y = \dfrac{1}{2}x + 1 \right\}$, the inverse is a function;

(b) $g = \{(x, y) \mid y = 2x^2\}$, $g^{-1} = \left\{ (x, y) \mid y = \pm\sqrt{\dfrac{x}{2}} \right\}$, the inverse is not a function
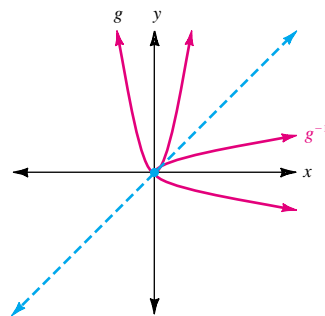
5. (a) Is one-to-one; **(b)** is not one-to-one

In exercises 1 to 6, write the inverse relation for each function. In each case, decide whether the inverse relation is also a function.

1. $\{(2, 3), (3, 4), (4, 5)\}$

2. $\{(2, 3), (3, 4), (4, 3)\}$

3. $\{(1, 2), (2, 2), (3, 2)\}$

4. $\{(5, 9), (3, 7), (7, 5)\}$

5. $\{(2, 4), (3, 9), (4, 16)\}$

6. $\{(-1, 2), (0, 3), (1, 2)\}$

In exercises 7 to 16, write an equation for the inverse of the relation defined by each equation.

7. $y = 2x + 8$

8. $y = -2x - 4$

9. $y = \dfrac{x - 1}{2}$

10. $y = \dfrac{x + 1}{3}$

11. $y = x^2 - 1$

12. $y = -x^2 + 2$

13. $x^2 + 4y^2 = 36$

14. $4x^2 + y^2 = 36$

15. $x^2 - y^2 = 9$

16. $4y^2 - x^2 = 4$

In exercises 17 to 22, write an equation for the inverse of the relation defined by each of the following, and graph the relation and its inverse on the same set of axes. Determine which inverse relations are also functions.

17. $y = 3x - 6$
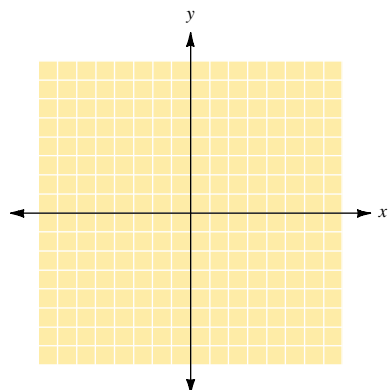
ANSWERS

1. _____

2. _____

3. _____

4. _____

5. _____

6. _____

7. _____

8. _____

9. _____

10. _____

11. _____

12. _____

13. _____

14. _____

15. _____

16. _____

17. _____

18. $y = 4x + 8$

19. $2x - 3y = 6$

20. $y = 3$

21. $y = x^2 + 1$

ANSWERS

21. _____

22. _____

23. _____

24. _____

25. _____

26. _____

27. _____

28. _____

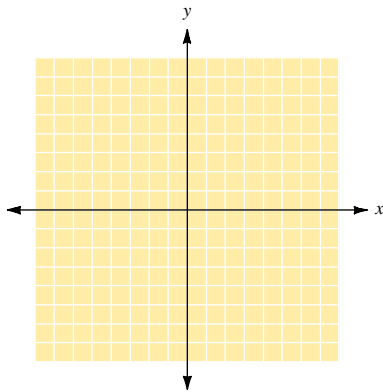29. _____

30. _____
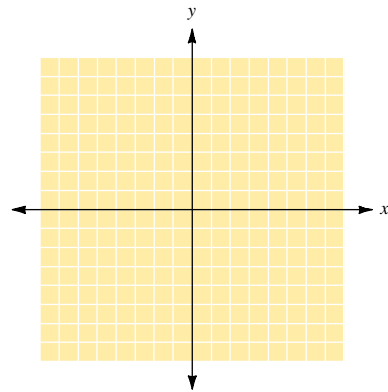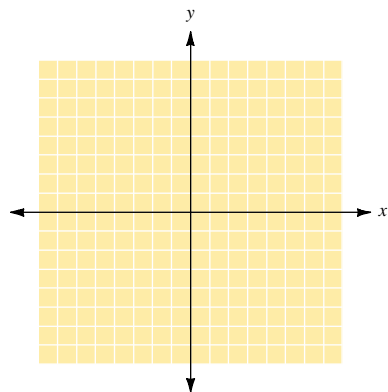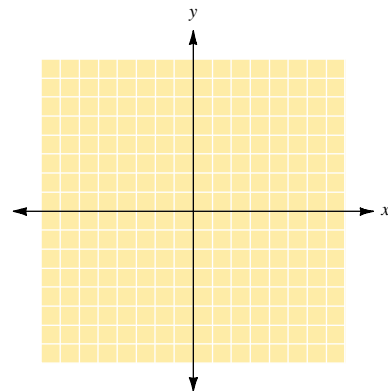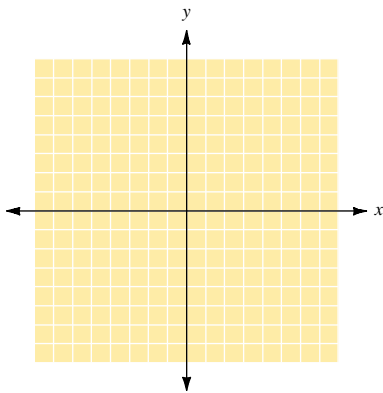
22. $y = -x^2 + 1$

23. An inverse process is an operation that undoes a procedure. If the procedure is wrapping a present, describe in detail the inverse process.

24. If the procedure is the series of steps that take you from home to your classroom, describe the inverse process.

If $f(x) = 3x - 6$, then $f^{-1}(x) = \dfrac{1}{3}x + 2$. Given these two functions, in exercises 25 to 30, find each of the following.

25. $f(6)$ **26.** $f^{-1}(6)$

27. $f(f^{-1}(6))$ **28.** $f^{-1}(f(6))$

29. $f(f^{-1}(x))$ **30.** $f^{-1}(f(x))$

If $g(x) = \dfrac{x + 1}{2}$, then $g^{-1}(x) = 2x - 1$. Given these two functions, in exercises 31 to 36, find each of the following.
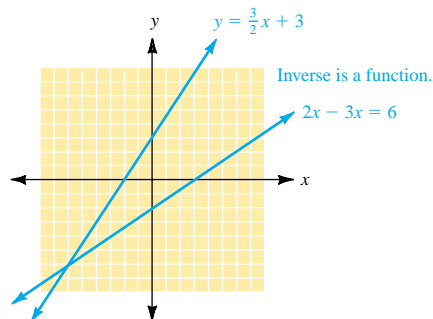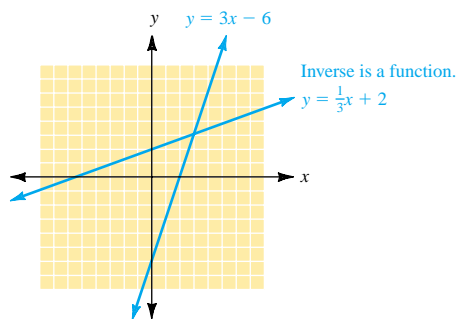
31. $g(3)$

32. $g^{-1}(3)$

33. $g(g^{-1}(3))$

34. $g^{-1}(g(3))$

35. $g(g^{-1}(x))$

36. $g^{-1}(g(x))$

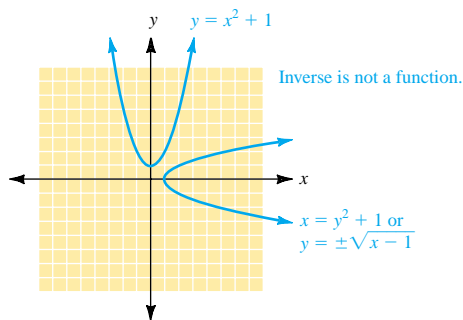Given $h(x) = 2x + 8$, then $h^{-1}(x) = \dfrac{x - 8}{2}$ in exercises 37 to 42, find each of the following.

37. $h(4)$

38. $h^{-1}(4)$

39. $h(h^{-1}(4))$

40. $h^{-1}(h(4))$

41. $h(h^{-1}(x))$

42. $h^{-1}(h(x))$

Answers

1. $\{(3, 2), (4, 3), (5, 4)\}$; function

3. $\{(2, 1), (2, 2), (2, 3)\}$; not a function

5. $\{(4, 2), (9, 3), (16, 4)\}$; function

7. $y = \dfrac{1}{2}x - 4$

9. $y = 2x + 1$

11. $y = \pm\sqrt{x + 1}$ or $x = y^2 - 1$

13. $4x^2 + y^2 = 36$

15. $y^2 - x^2 = 9$

17. Inverse is a function

19. Inverse is a function

21. Inverse is not a function

23.   **25.** 12   **27.** 6   **29.** x **31.** 2   **33.** 3   **35.** x

37. 16   **39.** 4   **41.** x