Factoring Trinomials

Section 5.5

Concepts

1. Factoring Trinomials: AC-Method

In Section 5.4, we learned how to factor out the greatest common factor from a polynomial and how to factor a four-term polynomial by grouping. In this section we present two methods to factor trinomials. The first method is called the ac-method. The second method is called the trial-and-error method.

The product of two binomials results in a four-term expression that can sometimes be simplified to a trinomial. To factor the trinomial, we want to reverse the process.

$$\text{Multiply:}\quad (2x+3)(x+2) \xrightarrow{\text{Multiply the binomials.}} 2x^2 + 4x + 3x + 6$$
$$\xrightarrow{\text{Add the middle terms.}} 2x^2 + 7x + 6$$

$$\text{Factor:}\quad 2x^2 + 7x + 6 \xrightarrow{\text{Rewrite the middle term as a sum or difference of terms.}} 2x^2 + 4x + 3x + 6$$
$$\xrightarrow{\text{Factor by grouping.}} (2x+3)(x+2)$$

To factor a trinomial $ax^2 + bx + c$ by the ac-method, we rewrite the middle term bx as a sum or difference of terms. The goal is to produce a four-term polynomial that can be factored by grouping. The process is outlined as follows.

The AC-Method to Factor $ax^2 + bx + c$ $(a \neq 0)$

1. Multiply the coefficients of the first and last terms, ac.
2. Find two integers whose product is ac and whose sum is b. (If no pair of integers can be found, then the trinomial cannot be factored further and is called a **prime polynomial**.)
3. Rewrite the middle term bx as the sum of two terms whose coefficients are the integers found in step 2.
4. Factor by grouping.

The ac-method for factoring trinomials is illustrated in Example 1. Before we begin, however, keep these two important guidelines in mind.

- For any factoring problem you encounter, always factor out the GCF from all terms first.
- To factor a trinomial, write the trinomial in the form $ax^2 + bx + c$.

Example 1 Factoring a Trinomial by the AC-Method

Factor. $12x^2 - 5x - 2$

Solution:

$12x^2 - 5x - 2$ — The GCF is 1.

$a = 12 \quad b = -5 \quad c = -2$ — **Step 1:** The expression is written in the form $ax^2 + bx + c$. Find the product $ac = 12(-2) = -24$.

Factors of –24	Factors of –24
(1)(−24)	(−1)(24)
(2)(−12)	(−2)(12)
(3)(−8)	(−3)(8)
(4)(−6)	(−4)(6)

Step 2: List all the factors of −24, and find the pair whose sum equals −5.
The numbers 3 and −8 produce a product of −24 and a sum of −5.

$12x^2 - 5x - 2$

$= 12x^2 + 3x - 8x - 2$

Step 3: Write the middle term of the trinomial as two terms whose coefficients are the selected numbers 3 and −8.

$= 12x^2 + 3x \;\vdots\; - 8x - 2$ — **Step 4:** Factor by grouping.

$= 3x(4x + 1) - 2(4x + 1)$

$= (4x + 1)(3x - 2)$ — The check is left for the reader.

Skill Practice

1. Factor $10x^2 + x - 3$.

Skill Practice Answers

1. $(5x + 3)(2x - 1)$

TIP: One frequently asked question is whether the order matters when we rewrite the middle term of the trinomial as two terms (step 3). The answer is no. From Example 1, the two middle terms in step 3 could have been reversed.

$$\begin{aligned} 12x^2 - 5x - 2 &= 12x^2 - 8x + 3x - 2 \\ &= 4x(3x - 2) + 1(3x - 2) \\ &= (3x - 2)(4x + 1) \end{aligned}$$

This example also shows that the order in which two factors are written does not matter. The expression $(3x - 2)(4x + 1)$ is equivalent to $(4x + 1)(3x - 2)$ because multiplication is a commutative operation.

Example 2 Factoring a Trinomial by the AC-Method

Factor the trinomial by using the ac-method. $-20c^3 + 34c^2d - 6cd^2$

Solution:

$-20c^3 + 34c^2d - 6cd^2$

$= -2c(10c^2 - 17cd + 3d^2)$ Factor out $-2c$.

Step 1: Find the product $a \cdot c = (10)(3) = 30$

Factors of 30	Factors of 30
$1 \cdot 30$	$(-1)(-30)$
$2 \cdot 15$	$(-2)(-15)$
$5 \cdot 6$	$(-5)(-6)$

Step 2: The numbers -2 and -15 form a product of 30 and a sum of -17.

$= -2c(10c^2 - 17cd + 3d^2)$

Step 3: Write the middle term of the trinomial as two terms whose coefficients are -2 and -15.

$= -2c(10c^2 - 2cd \mid - 15cd + 3d^2)$ **Step 4:** Factor by grouping.

$= -2c[2c(5c - d) - 3d(5c - d)]$

$= -2c(5c - d)(2c - 3d)$

Skill Practice Factor by the ac-method.

2. $-4wz^3 - 2w^2z^2 + 20w^3z$

TIP: In Example 2, removing the GCF from the original trinomial produced a new trinomial with smaller coefficients. This makes the factoring process simpler because the product ac is smaller.

Original trinomial	With the GCF factored out
$-20c^3 + 34c^2d - 6cd^2$	$-2c(10c^2 - 17cd + 3d^2)$
$ac = (-20)(-6) = 120$	$ac = (10)(3) = 30$

Skill Practice Answers

2. $-2wz(2z + 5w)(z - 2w)$

2. Factoring Trinomials: Trial-and-Error Method

Another method that is widely used to factor trinomials of the form $ax^2 + bx + c$ is the trial-and-error method. To understand how the trial-and-error method works, first consider the multiplication of two binomials:

Product of $2 \cdot 1$ Product of $3 \cdot 2$

$$(2x + 3)(1x + 2) = 2x^2 + \underbrace{4x + 3x}_{\substack{\text{sum of products}\\\text{of inner terms}\\\text{and outer terms}}} + 6 = 2x^2 + 7x + 6$$

To factor the trinomial $2x^2 + 7x + 6$, this operation is reversed. Hence

Factors of 2

$$2x^2 + 7x + 6 = (\square x \quad \square)(\square x \quad \square)$$

Factors of 6

We need to fill in the blanks so that the product of the first terms in the binomials is $2x^2$ and the product of the last terms in the binomials is 6. Furthermore, the factors of $2x^2$ and 6 must be chosen so that the sum of the products of the inner terms and outer terms equals $7x$.

To produce the product $2x^2$, we might try the factors $2x$ and x within the binomials.

$$(2x \quad \square)(x \quad \square)$$

To produce a product of 6, the remaining terms in the binomials must either both be positive or both be negative. To produce a positive middle term, we will try positive factors of 6 in the remaining blanks until the correct product is found. The possibilities are $1 \cdot 6, 2 \cdot 3, 3 \cdot 2$, and $6 \cdot 1$.

$(2x + 1)(x + 6) = 2x^2 + 12x + 1x + 6 = 2x^2 + 13x + 6$	Wrong middle term
$(2x + 2)(x + 3) = 2x^2 + 6x + 2x + 6 = 2x^2 + 8x + 6$	Wrong middle term
$\mathbf{(2x + 3)(x + 2)} = 2x^2 + 4x + 3x + 6 = \mathbf{2x^2 + 7x + 6}$	Correct!
$(2x + 6)(x + 1) = 2x^2 + 2x + 6x + 6 = 2x^2 + 8x + 6$	Wrong middle term

The correct factorization of $2x^2 + 7x + 6$ is $(2x + 3)(x + 2)$. ✔

As this example shows, we factor a trinomial of the form $ax^2 + bx + c$ by shuffling the factors of a and c within the binomials until the correct product is obtained. However, sometimes it is not necessary to test all the possible combinations of factors. In this example, the GCF of the original trinomial is 1. Therefore, any binomial factor that shares a common factor *greater than 1* does not need to be considered. In this case the possibilities $(2x + 2)(x + 3)$ and $(2x + 6)(x + 1)$ cannot work.

$$\underbrace{(2x + 2)}_{\substack{\text{Common}\\\text{factor of 2}}}(x + 3) \qquad \underbrace{(2x + 6)}_{\substack{\text{Common}\\\text{factor of 2}}}(x + 1)$$

The steps to factor a trinomial by the trial-and-error method are outlined as follows.

The Trial-and-Error Method to Factor $ax^2 + bx + c$

1. Factor out the greatest common factor.
2. List all pairs of positive factors of a and pairs of positive factors of c. Consider the reverse order for either list of factors.
3. Construct two binomials of the form

Factors of a

$$(\square x \quad \square)(\square x \quad \square)$$

Factors of c

Test each combination of factors and signs until the correct product is found. If no combination of factors produces the correct product, the trinomial cannot be factored further and is a **prime polynomial**.

Example 3 Factoring a Trinomial by the Trial-and-Error Method

Factor the trinomial by the trial-and-error method. $10x^2 - 9x - 1$

Solution:

$10x^2 - 9x - 1$ **Step 1:** Factor out the GCF from all terms. The GCF is 1. The trinomial is written in the form $ax^2 + bx + c$.

To factor $10x^2 - 9x - 1$, two binomials must be constructed in the form

Factors of 10

$$(\square x \quad \square)(\square x \quad \square)$$

Factors of -1

Step 2: To produce the product $10x^2$, we might try $5x$ and $2x$ or $10x$ and $1x$. To produce a product of -1, we will try the factors $1(-1)$ and $-1(1)$.

Step 3: Construct all possible binomial factors, using different combinations of the factors of $10x^2$ and -1.

$(5x + 1)(2x - 1) = 10x^2 - 5x + 2x - 1 = 10x^2 - 3x - 1$ Wrong middle term

$(5x - 1)(2x + 1) = 10x^2 + 5x - 2x - 1 = 10x^2 + 3x - 1$ Wrong middle term

The numbers 1 and -1 did not produce the correct trinomial when coupled with $5x$ and $2x$, so we try $10x$ and $1x$.

$(10x - 1)(1x + 1) = 10x^2 + 10x - 1x - 1 = 10x^2 + 9x - 1$ Wrong middle term

$\mathbf{(10x + 1)(1x - 1)} = 10x^2 - 10x + 1x - 1 = \mathbf{10x^2 - 9x - 1}$ Correct!

Hence $10x^2 - 9x - 1 = (10x + 1)(x - 1)$

Skill Practice Factor by trial and error.

3. $5y^2 - 9y + 4$

Skill Practice Answers

3. $(5y - 4)(y - 1)$

In Example 3, the factors of -1 must have opposite signs to produce a negative product. Therefore, one binomial factor is a sum and one is a difference. Determining the correct signs is an important aspect of factoring trinomials. We suggest the following guidelines:

TIP: Given the trinomial $ax^2 + bx + c$ ($a > 0$), the signs can be determined as follows:

1. If c is *positive*, then the signs in the binomials must be the same (either both positive or both negative). The correct choice is determined by the middle term. If the middle term is positive, then both signs must be positive. If the middle term is negative, then both signs must be negative.

   Example: $20x^2 + 43x + 21$ (c is positive.)
   $(4x + 3)(5x + 7)$ same signs

   Example: $20x^2 - 43x + 21$ (c is positive.)
   $(4x - 3)(5x - 7)$ same signs

2. If c is *negative*, then the signs in the binomials must be different. The middle term in the trinomial determines which factor gets the positive sign and which factor gets the negative sign.

   Example: $x^2 + 3x - 28$ (c is negative.)
   $(x + 7)(x - 4)$ different signs

   Example: $x^2 - 3x - 28$ (c is negative.)
   $(x - 7)(x + 4)$ different signs

Example 4 Factoring a Trinomial

Factor the trinomial by the trial-and-error method. $8y^2 + 13y - 6$

Solution:

$8y^2 + 13y - 6$ **Step 1:** The GCF is 1.

$(\square y \quad \square)(\square y \quad \square)$

Factors of 8	Factors of 6	
$1 \cdot 8$	$1 \cdot 6$	
$2 \cdot 4$	$2 \cdot 3$	
	$3 \cdot 2$	(reverse order)
	$6 \cdot 1$	(reverse order)

Step 2: List the positive factors of 8 and positive factors of 6. Consider the reverse order in one list of factors.

Step 3: Construct all possible binomial factors by using different combinations of the factors of 8 and 6.

$(2y \quad 1)(4y \quad 6)$
$(2y \quad 2)(4y \quad 3)$
$(2y \quad 3)(4y \quad 2)$
$(2y \quad 6)(4y \quad 1)$
$(1y \quad 1)(8y \quad 6)$
$(1y \quad 3)(8y \quad 2)$

Without regard to signs, these factorizations cannot work because the terms in the binomial share a common factor greater than 1.

Test the remaining factorizations. Keep in mind that to produce a product of -6, the signs within the parentheses must be opposite (one positive and one negative). Also, the sum of the products of the inner terms and outer terms must be combined to form $13y$.

$(1y \quad 6)(8y \quad 1)$ *Incorrect.* Wrong middle term.

Regardless of signs, the product of inner terms $48y$ and the product of outer terms $1y$ cannot be combined to form the middle term $13y$.

$(1y \quad 2)(8y \quad 3)$ *Correct.* The terms $16y$ and $3y$ can be combined to form the middle term $13y$, provided the signs are applied correctly. We require $+16y$ and $-3y$.

Hence, the correct factorization of $8y^2 + 13y - 6$ is $(y + 2)(8y - 3)$.

Skill Practice Factor by trial-and-error.

4. $4t^2 + 5t - 6$

Example 5 Factoring a Trinomial by the Trial-and-Error Method

Factor the trinomial by the trial-and-error method.

$$-80x^3y + 208x^2y^2 - 20xy^3$$

Solution:

$-80x^3y + 208x^2y^2 - 20xy^3$

$= -4xy(20x^2 - 52xy + 5y^2)$ **Step 1:** Factor out $-4xy$.

$= -4xy(\square x \quad \square y)(\square x \quad \square y)$

Factors of 20	Factors of 5
$1 \cdot 20$	$1 \cdot 5$
$2 \cdot 10$	$5 \cdot 1$
$4 \cdot 5$	

Step 2: List the positive factors of 20 and positive factors of 5. Consider the reverse order in one list of factors.

Step 3: Construct all possible binomial factors by using different combinations of the factors of 20 and factors of 5. The signs in the parentheses must both be negative.

$-4xy(1x - 1y)(20x - 5y)$
$-4xy(2x - 1y)(10x - 5y)$
$-4xy(4x - 1y)(5x - 5y)$

Incorrect. These binomials contain a common factor.

Skill Practice Answers

4. $(4t - 3)(t + 2)$

$-4xy(1x - 5y)(20x - 1y)$	*Incorrect.*	Wrong middle term. $-4xy(x - 5y)(20x - 1y)$ $= -4xy(20x^2 - 101xy + 5y^2)$
$-4xy(2x - 5y)(10x - 1y)$	***Correct.***	$-4xy(2x - 5y)(10x - 1y)$ $= \mathbf{-4xy(20x^2 - 52xy + 5y^2)}$ $= -80x^3y + 208x^2y^2 - 20xy^3$
$-4xy(4x - 5y)(5x - 1y)$	*Incorrect.*	Wrong middle term. $-4xy(4x - 5y)(5x - 1y)$ $= -4xy(20x^2 - 29x + 5y^2)$

The correct factorization of $-80x^3y + 208x^2y^2 - 20xy^3$ is $-4xy(2x - 5y)(10x - y)$.

Skill Practice Factor by the trial-and-error method.

5. $-4z^3 - 22z^2 - 30z$

3. Factoring Trinomials with a Leading Coefficient of 1

If a trinomial has a leading coefficient of 1, the factoring process simplifies significantly. Consider the trinomial $x^2 + bx + c$. To produce a leading term of x^2, we can construct binomials of the form $(x + \square)(x + \square)$. The remaining terms may be satisfied by two numbers p and q whose product is c and whose sum is b:

$$(x + \overbrace{p)(x + q}^{\text{Factors of } c}) = x^2 + qx + px + pq = x^2 + \underbrace{(p + q)}_{\text{Sum } = b}x + \underbrace{pq}_{\text{Product } = c}$$

This process is demonstrated in Example 6.

Example 6 **Factoring a Trinomial with a Leading Coefficient of 1**

Factor the trinomial.

$$x^2 - 10x + 16$$

Solution:

$x^2 - 10x + 16$	Factor out the GCF from all terms. In this case, the GCF is 1.
$= (x \quad \square)(x \quad \square)$	The trinomial is written in the form $x^2 + bx + c$. To form the product x^2, use the factors x and x.

Skill Practice Answers

5. $-2z(2z + 5)(z + 3)$

Next, look for two numbers whose product is 16 and whose sum is -10. Because the middle term is negative, we will consider only the negative factors of 16.

Factors of 16	**Sum**
$-1(-16)$	$-1 + (-16) = -17$
$\mathbf{-2(-8)}$	$\mathbf{-2 + (-8) = -10}$
$-4(-4)$	$-4 + (-4) = -8$

The numbers are -2 and -8.

Hence $x^2 - 10x + 16 = (x - 2)(x - 8)$

Skill Practice Factor.

6. $c^2 + 6c - 27$

4. Factoring Perfect Square Trinomials

Recall from Section 5.2 that the square of a binomial always results in a **perfect square trinomial**.

$$(a + b)^2 = (a + b)(a + b) = a^2 + ab + ab + b^2 = a^2 + 2ab + b^2$$

$$(a - b)^2 = (a - b)(a - b) = a^2 - ab - ab + b^2 = a^2 - 2ab + b^2$$

For example, $(2x + 7)^2 = (2x)^2 + 2(2x)(7) + (7)^2 = 4x^2 + 28x + 49$

$a = 2x \quad b = 7 \qquad\qquad a^2 + 2ab + b^2$

To factor the trinomial $4x^2 + 28x + 49$, the ac-method or the trial-and-error method can be used. However, recognizing that the trinomial is a perfect square trinomial, we can use one of the following patterns to reach a quick solution.

Factored Form of a Perfect Square Trinomial

$$a^2 + 2ab + b^2 = (a + b)^2$$

$$a^2 - 2ab + b^2 = (a - b)^2$$

TIP: To determine if a trinomial is a perfect square trinomial, follow these steps:

1. Check if the first and third terms are both perfect squares with positive coefficients.
2. If this is the case, identify *a* and *b*, and determine if the middle term equals 2*ab*.

Example 7 Factoring Perfect Square Trinomials

Factor the trinomials completely.

a. $x^2 + 12x + 36$ **b.** $4x^2 - 36xy + 81y^2$

Skill Practice Answers

6. $(c + 9)(c - 3)$

Solution:

a. $x^2 + 12x + 36$

The GCF is 1.

Perfect squares

$= x^2 + 12x + 36$

- The first and third terms are positive.
- The first term is a perfect square: $x^2 = (x)^2$
- The third term is a perfect square: $36 = (6)^2$
- The middle term is twice the product of x and 6:

$$12x = 2(x)(6)$$

$= (x)^2 + 2(x)(6) + (6)^2$

Hence the trinomial is in the form $a^2 + 2ab + b^2$, where $a = x$ and $b = 6$.

$= (x + 6)^2$

Factor as $(a + b)^2$.

b. $4x^2 - 36xy + 81y^2$

The GCF is 1.

Perfect squares

$= 4x^2 - 36xy + 81y^2$

- The first and third terms are positive.
- The first term is a perfect square: $4x^2 = (2x)^2$.
- The third term is a perfect square: $81y^2 = (9y)^2$.
- The middle term:

$$-36xy = -2(2x)(9y)$$

$= (2x)^2 - 2(2x)(9y) + (9y)^2$

The trinomial is in the form $a^2 - 2ab + b^2$, where $a = 2x$ and $b = 9y$.

$= (2x - 9y)^2$

Factor as $(a - b)^2$.

Skill Practice Factor completely.

7. $x^2 + 2x + 1$ **8.** $9y^2 - 12yz + 4z^2$

5. Mixed Practice: Summary of Factoring Trinomials

Summary: Factoring Trinomials of the Form $ax^2 + bx + c$ $(a \neq 0)$

When factoring trinomials, the following guidelines should be considered:

1. Factor out the greatest common factor.
2. Check to see if the trinomial is a perfect square trinomial. If so, factor it as either $(a + b)^2$ or $(a - b)^2$. (With a perfect square trinomial, you do not need to use the ac-method or trial-and-error method.)
3. If the trinomial is not a perfect square, use either the ac-method or the trial-and-error method to factor.
4. Check the factorization by multiplication.

Skill Practice Answers

7. $(x + 1)^2$ **8.** $(3y - 2z)^2$

Example 8 Factoring Trinomials

Factor the trinomials completely.

a. $80s^3t + 80s^2t^2 + 20st^3$ **b.** $5w^2 + 50w + 45$ **c.** $2p^2 + 9p + 14$

Solution:

a. $80s^3t + 80s^2t^2 + 20st^3$

$= 20st(4s^2 + 4st + t^2)$ The GCF is $20st$.

Perfect squares

$= 20st(4s^2 + 4st + t^2)$

- The first and third terms are positive.
- The first and third terms are perfect squares: $4s^2 = (2s)^2$ and $t^2 = (t)^2$
- Because $4st = 2(2s)(t)$, the trinomial is in the form $a^2 + 2ab + b^2$, where $a = 2s$ and $b = t$.

$= 20st(2s + t)^2$ Factor as $(a + b)^2$.

b. $5w^2 + 50w + 45$

$= 5(w^2 + 10w + 9)$ The GCF is 5.

Perfect squares

$= 5(w^2 + 10w + 9)$

The first and third terms are perfect squares: $w^2 = (w)^2$ and $9 = (3)^2$.

However, the middle term $10w \neq 2(w)(3)$. Therefore, this is *not* a perfect square trinomial.

$= 5(w + 9)(w + 1)$ To factor, use either the ac-method or the trial-and-error method.

c. $2p^2 + 9p + 14$

The GCF is 1. The trinomial is not a perfect square trinomial because neither 2 nor 14 is a perfect square. Therefore, try factoring by either the ac-method or the trial-and-error method. We use the trial-and-error method here.

Factors of 2	**Factors of 14**
$2 \cdot 1$	$1 \cdot 14$
	$14 \cdot 1$
	$2 \cdot 7$
	$7 \cdot 2$

After constructing all factors of 2 and 14, we see that no combination of factors will produce the correct result.

$(2p + 14)(p + 1)$ *Incorrect*: $(2p + 14)$ contains a common factor of 2.

$(2p + 2)(p + 7)$ *Incorrect*: $(2p + 2)$ contains a common factor of 2.

$$(2p + 1)(p + 14) = 2p^2 + 28p + p + 14 \longrightarrow 2p^2 + 29p + 14 \quad \textit{Incorrect}$$
(wrong middle term)

$$(2p + 7)(p + 2) = 2p^2 + 4p + 7p + 14 \longrightarrow 2p^2 + 11p + 14 \quad \textit{Incorrect}$$
(wrong middle term)

Because none of the combinations of factors results in the correct product, we say that the trinomial $2p^2 + 9p + 14$ is prime. This polynomial cannot be factored by the techniques presented here.

Skill Practice Factor completely.

9. $-x^2 + 6x - 9$ **10.** $6v^2 - 12v - 18$ **11.** $6r^2 - 13rs + 10s^2$

Skill Practice Answers

9. $-(x - 3)^2$
10. $6(v + 1)(v - 3)$
11. Prime