

# 4 GRAPH TRANSFORMS

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## Objectives

After studying this chapter you should

- be able to use appropriate technology to investigate graphical transformations;
- understand how complicated functions can be built up from transformations of simple functions;
- be able to predict the graph of functions after various transformations.

## 4.0 Introduction

You have already seen that the ability to illustrate a function graphically is a very useful one. Graphs can easily be used to explain or predict, so it is important to be able to sketch quickly the main features of a graph of a function. New technology, particularly graphic calculators, provides very useful tools for finding shapes but as a mathematician you will still need to gain the ability to understand what effect various transformations have on the graph of a function. First try the activity below **without** using a graphic calculator or computer.

### Activity 1

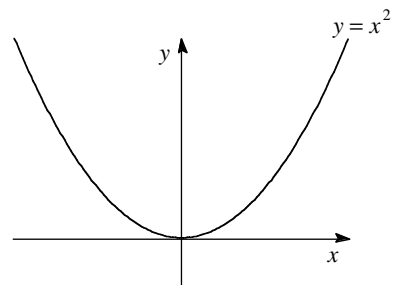
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You should be familiar with the graph of

$$y = x^2.$$

It is shown on the right. Without using any detailed calculations or technology, predict the shape of the graphs of the following

- (a)  $y = x^2 + 1$       (b)  $y = 2x^2$       (c)  $y = 3x^2$   
 (d)  $y = (x - 1)^2$       (e)  $y = (x + 1)^2$       (f)  $y = \frac{1}{x^2}$  ( $x \neq 0$ ).

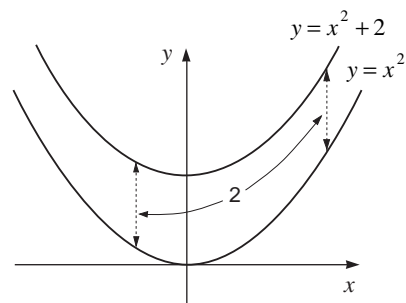


Now check your answers using a graphic calculator or computer.

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## 4.1 Transformation of axes

Suppose  $y = x^2$ , then the graph of  $y = x^2 + 2$  moves the curve up by two units as is shown in the figure opposite. For any  $x$  value, the  $y$  value will be increased by two units.



What does the graph of  $y = x^2 + a$  look like?

What you are doing in the example above is equivalent to moving the  $x$  axes down by 2 units, which you can see by defining

$$Y = y - 2.$$

Then  $Y = x^2$  and you are back to the original equation.

Describe the graph of  $y = x^3 + 1$

This type of transformation

$$f(x) \mapsto f(x) + a$$

is called a **translation** of the graph by  $a$  units along the  $y$ -axis.

### Example

Find the value of  $a$  so that

$$y = x^2 - 2x + a$$

just touches the  $x$ -axis.

### Solution

The graph of  $y = x^2 - 2x$  is shown opposite. From this, you can see that it needs to be raised one unit, since its minimum value of  $-1$  is obtained at  $x = 1$ . So the new equation will be

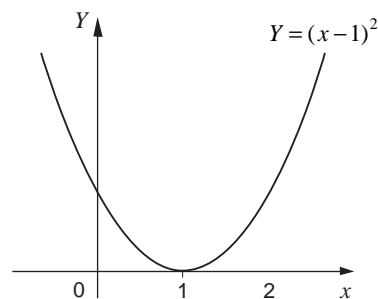
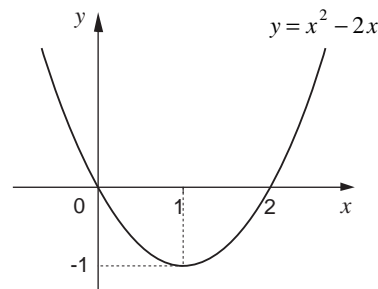
$$Y = x^2 - 2x + 1.$$

Note that the new  $Y$  function can be written as

$$Y = (x - 1)^2 \geq 0 \text{ for all } x$$

and equality only occurs when  $x = 1$  (as illustrated).

As well as translations along the  $y$ -axis, you can perform similar operations along the  $x$ -axis.



## Activity 2 Translations parallel to the $x$ -axis

Again use the familiar  $y = x^2$  curve, but this time write it as

$$f(x) = x^2.$$

Evaluate  $f(x-2)$ . Sketch the graph of  $y = f(x-2)$ .

What is the relationship between this curve and the original.

If you know the shape of  $y = f(x)$ , what does the graph of  $y = f(x-a)$  look like?

The transformation

$$f(x) \mapsto f(x-a)$$

is a translation by  $a$  units along the  $x$ -axis.

### Example

The function  $f(x)$  is defined by

$$f(x) = x^3 - 3x^2 + 3x - 1.$$

By considering  $y = f(x+1)$ , deduce the shape of the graph of  $f(x)$ .

### Solution

$$f(x+1) = (x+1)^3 - 3(x+1)^2 + 3(x+1) - 1$$

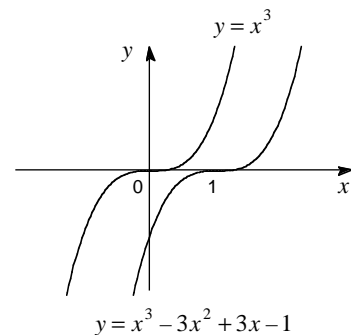
$$(x+1)^2 = (x+1)(x+1) = x^2 + 2x + 1$$

$$\begin{aligned} (x+1)^3 &= (x+1)(x+1)^2 \\ &= (x+1)(x^2 + 2x + 1) \\ &= x^3 + 3x^2 + 3x + 1 \end{aligned}$$

$$\begin{aligned} f(x+1) &= x^3 + 3x^2 + 3x + 1 \\ &\quad - 3(x^2 + 2x + 1) + 3(x+1) - 1 \\ &= x^3 + x^2(+3-3) + x(3-6+3) + 1-3+3-1 \\ &= x^3 \end{aligned}$$

Hence  $y = f(x+1) = x^3$  and this is illustrated opposite.

This means that  $f(x)$  must also have this shape, but moved one unit along the  $x$ -axis.



### Activity 3

Using a graphic calculator or computer,

- (a) illustrate the curves

$$y = x^3 \text{ and } y = x^3 - 3x^2 + 3x - 1$$

and hence verify the result in the sketch on the previous page;

- (b) illustrate the curves

$$y = x^4 \text{ and } y = x^4 - 8x^3 + 24x^2 - 32x + 16$$

and deduce a simpler form to write the second function.

Use  $x$  range  $-2$  to  $4$  and  $y$  range  $0$  to  $10$ .

### Exercise 4A

1. Without using a graph plotting device, draw sketches of

$$f(x), f(x+5), f(x)+5$$

for the following functions

(a)  $f(x) = 2x - 1$

(b)  $f(x) = x^2 - 1$

(c)  $f(x) = (x-1)^2$ .

2. Use a graph plotting device to illustrate the graphs of

$$f(x) = x^2, g(x) = x^2 + 2x + 2.$$

Hence or otherwise write  $g(x)$  in the form

$$f(x+a)+b$$

by finding the constants  $a$  and  $b$ .

3. If  $f(x) = \frac{1}{x}$ , sketch the graphs of

(a)  $f(x)$       (b)  $f(x-1)$       (c)  $f(x-1)+1$ .

## 4.2 Stretches

In this section you will be investigating the effect of stretching either the  $y$ - or  $x$ -axis.

### Example

For the function

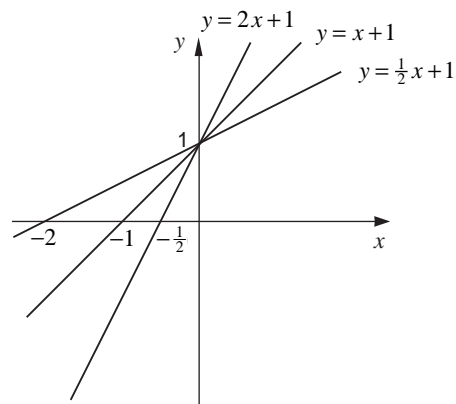
$$y = f(x) = x + 1$$

draw the graphs of

(a)  $f(2x)$     (b)  $f(\frac{1}{2}x)$     (c)  $2f(x)$     (d)  $\frac{1}{2}f(x)$ .

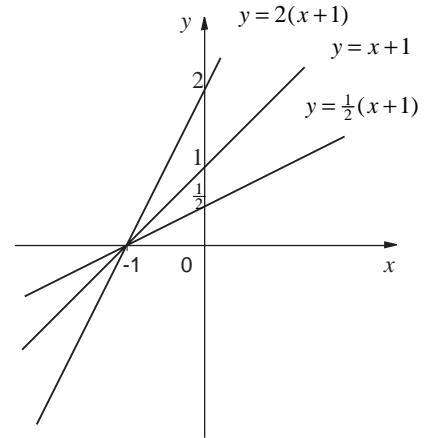
### Solution

(a)  $f(2x) = 2x + 1$  } these are illustrated opposite  
 (b)  $f(\frac{1}{2}x) = \frac{1}{2}x + 1$  }



- (c)  $2f(x) = 2(x+1)$  } again illustrated opposite  
 (d)  $\frac{1}{2}f(x) = \frac{1}{2}(x+1)$  }

You should be beginning to get a feel for what the various types of transformations do, and the next activity will help you to clarify your ideas.



### Activity 4 Stretches

For the function

$$y = f(x) = x^2 + 1$$

draw the graphs of

- (a)  $2f(x)$  (b)  $\frac{1}{2}f(x)$  (c)  $f(2x)$  (d)  $f(\frac{1}{2}x)$ .

Use a graph plotting device to help you if you are not sure of what the graphs look like.

The example and the activity have shown you that

$y = \alpha f(x)$  is a stretch, parallel to the  $y$ -axis, by a factor  $\alpha$   
 $y = f(\alpha x)$  is a stretch, parallel to the  $x$ -axis, by a factor  $\frac{1}{\alpha}$

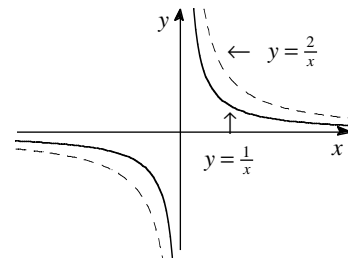
### Example

If  $f(x) = \frac{1}{x}$ , illustrate (a)  $2f(x)$  (b)  $f(\frac{1}{2}x)$ .

#### Solution

(a)  $2f(x) = \frac{2}{x}$ ; this is illustrated opposite.

(b)  $f(\frac{1}{2}x) = \frac{1}{(\frac{1}{2}x)} = \frac{2}{x}$ , which is identical to  $2f(x)$ .



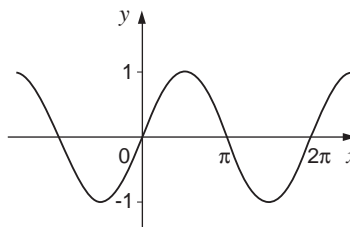
For this rather special function, a stretch of factor  $\alpha$  parallel to the  $y$ -axis is identical to a stretch of factor  $\alpha$  parallel to the  $x$ -axis.

Why are the two transformations identical for the function

$$y = \frac{1}{x} ?$$

### Exercise 4B

- For the function  $f(x) = 2x - 1$  illustrate the graphs of
  - $f(\frac{1}{2}x)$
  - $f(2x)$
  - $2f(x)$
  - $\frac{1}{2}f(x)$ .
- For which of the following does the function  $y = f(x)$  remain unaltered by the transformation  $y = \frac{1}{\alpha}f(\alpha x)$ ?
  - $f(x) = x$
  - $f(x) = x + 1$
  - $f(x) = x^2$
  - $f(x) = \frac{2}{x}$ .
- For the function  $y = f(x)$ , shown below, sketch the curves defined by
  - $y = f(\frac{1}{2}x)$
  - $y = 2f(x)$ .



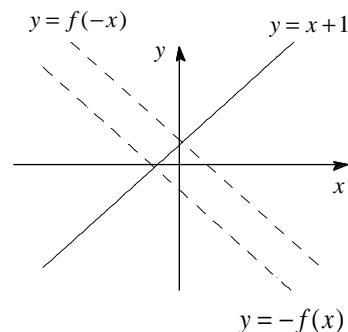
## 4.3 Reflections

If  $f(x) = x + 1$ , then the graph of  $-f(x) = -(x + 1)$  is seen to be a **reflection** in the  $x$ -axis.

On the other hand

$$f(-x) = -x + 1$$

can be seen to be a **reflection** in the  $y$ -axis.



### Activity 5 Reflections

For each of the functions below sketch (i)  $-f(x)$  (ii)  $f(-x)$ :

- $f(x) = x^2$
- $f(x) = 2x + 1$
- $f(x) = x^3$
- $f(x) = \frac{1}{x}$ .

Use a graphic calculator or computer to check your answers if you have any doubt.

You have seen that

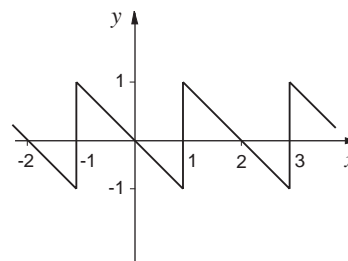
$f(-x)$  is a reflection in the  $y$ -axis  
 $-f(x)$  is a reflection in the  $x$ -axis

It is now possible to combine various transformations.

### Example

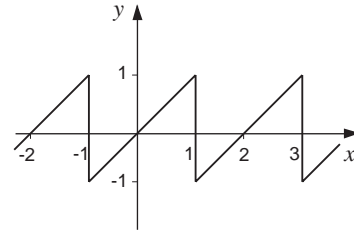
If the graph of  $y = f(x)$  is shown opposite, illustrate the shape of

$$y = 2f(-x) + 3.$$



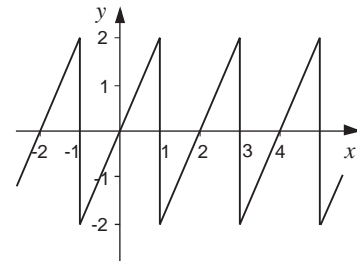
**Solution**

To find  $f(-x)$ , you reflect in the  $y$ -axis to give the graph opposite.



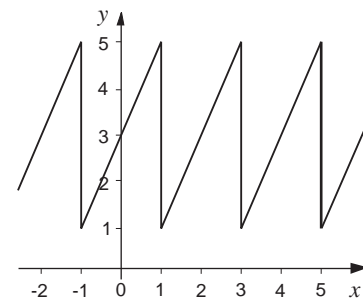
The sketch of  $2f(-x)$  is shown opposite.

This is a stretch of factor 2 along the  $y$ -axis.



Finally adding 3 to each value gives the graph shown opposite,

$$y = 2f(-x) + 3.$$

**Example**

Sketch the graph of  $y = \frac{2}{x-1} + 2$ .

**Solution**

Of course, you could find its graph very quickly using a graphic calculator or computer. It is, though, instructive to build up the sketch starting from a simple function, say

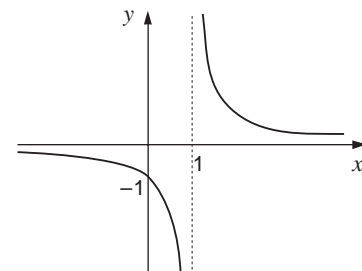
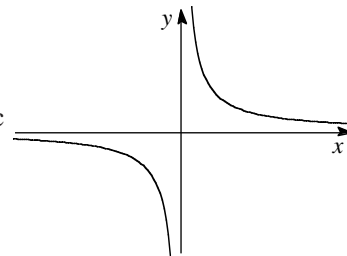
$$f(x) = \frac{1}{x}$$

and performing transformations to obtain the required function.

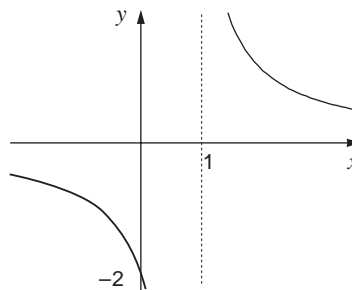
In terms of  $f$ , you can write

$$y = 2f(x-1) + 2.$$

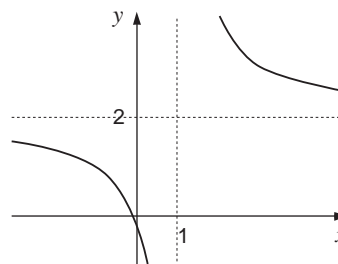
So you must first sketch  $y = f(x-1)$  as shown opposite.



Now sketch  $y = 2f(x - 1)$  as shown opposite.



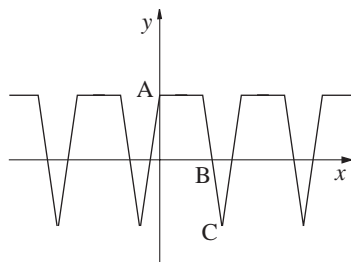
Finally you add 2 to the function to give the sketch opposite.



The ability to sketch curves quickly will be very useful throughout your course of study of mathematics. Although modern technology does make it much easier to find graphs, the process of **understanding** both what various transformations do and how more complex functions can be built up from a simple function is crucial for becoming a competent mathematician.

### Exercise 4C

1. The graph below is a sketch of  $y = f(x)$ , showing three points A, B and C.



Sketch a graph of the following functions:

- (a)  $f(-x)$     (b)  $2f(x)$     (c)  $f\left(\frac{x}{3}\right)$   
 (d)  $-\frac{1}{2}f(x)$     (e)  $f(x+3)$     (f)  $2f(x+1)$   
 (g)  $f(x)+5$ .

In each case indicate the position of A, B and C on the transformed graphs.

2. Using the functions  $f(x) = x^2$ ,  $g(x) = \frac{1}{x}$

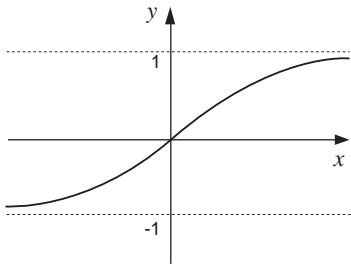
show how each of the following functions can be expressed in terms of  $f$  or  $g$ . Hence sketch these graphs.

- (a)  $y = 2x^2 + 1$     (b)  $y = 4 - x^2$   
 (c)  $y = \frac{1}{(x+4)} + 2$  ( $x \neq -4$ )    (d)  $y = -\frac{2}{x} + 1$  ( $x \neq 0$ )  
 (e)  $y = x^2 + 2x + 4$ .



## 4.4 Miscellaneous Exercises

1. The function  $y = f(x)$  is illustrated below.



Sketch the following functions:

- (a)  $y = -f(x)$     (b)  $y = f(-x)$   
 (c)  $y = f(x) + 1$     (d)  $y = 2f(x)$ .

2. Express each of the following functions in terms of either  $f(x) = x^2$  or  $g(x) = \frac{1}{x}$ .

(a)  $y(x) = 4x^2 + 1$

(b)  $y(x) = 1 - \frac{1}{(x+1)}$  ( $x \neq -1$ ).

3. Sketch the graph of

$$f(x) = \frac{1}{x^3} \quad (x \neq 0).$$

Show that

$$f(-x) = -f(x).$$

What does this tell you about the function?

