# SPECIALIST MATHEMATICS

# SCAT 1: PROBLEM SOLVING

# PROBLEM 2: HYPERBOLIC FUNCTIONS

**CONTENTS**

# QUESTION 1- SKETCH GRAPHS OF COSH AND SINH X.

## SKETCH GRAPH OF COSH X:

let $y = \cosh x$

$\cosh x = \frac{1}{2}(e^x + e^{-x})$

$\therefore y = \frac{1}{2}(e^x + e^{-x})$

## X, Y INTERCEPTS

$x$-intercept at $y = 0$

$\therefore \quad 0 = \frac{1}{2}(e^x + e^{-x})$

$0 = e^x + e^{-x}$ but $e^x > 0$ and $e^{-x} > 0$, $\therefore$ **no x-intercept.**

$y$-intercept at $x = 0$

$y = \frac{1}{2}(e^0 + e^0)$

$= 1.$

**$\therefore$ $y$-intercept at point (0,1).**

## ASYMPTOTES

$\lim_{x\to\infty} \quad (e^x + e^{-x}) = \infty$

$\lim_{x\to-\infty} \quad (e^x + e^{-x}) = \infty$

**no horizontal asymptotes.** As $y$ is defined for all real $x$, there are **no vertical asymptotes.**

## TURNING POINTS

occur when $\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{1}{2}(e^x - e^{-x})$

$0 = \frac{1}{2}(e^x - e^{-x})$

$= e^x - e^{-x}$

$\therefore e^x = e^{-x}$

$\therefore -x = x$

$-2x = 0$

$\therefore x = 0$

when $x > 0$, eg. $x = 1$

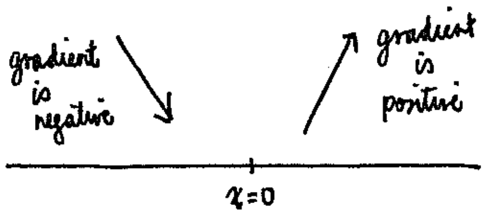
$\frac{dy}{dx} = \frac{1}{2}(e^{1} - e^{1}) = 1.18$

this is positive, $\therefore$ gradient positive.

when $x < 0$, eg. $x = -1$

$\frac{dy}{dx} = \frac{1}{2}(e^{-1} - e^{1}) = -1.18$

this is negative, $\therefore$ gradient negative.




$\therefore$ using First Derivative Test, local minimum occurs when $x = 0$.

$x = 0$,

$\therefore y = \frac{1}{2}(e^{0} + e^{0}) = 1$

$\therefore$ **graph turns at (0,1)**

CONCAVITY/POINTS OF INFLECTION

determined by second derivative of function. Inflection points occur when $\frac{d^2y}{dx^2} = 0$

$\therefore \frac{d^2y}{dx^2} = \frac{1}{2}(e^{x} + e^{-x})$

and $\frac{d^2y}{dx^2} = 0$

$\therefore 0 = \frac{1}{2}(e^{x} + e^{-x})$

$\therefore 0 = e^{x} + e^{-x}$ but impossible, $\therefore$ **no inflection points.**

Concavity- determined by value of $\frac{d^2y}{dx^2}$.

When $\frac{d^2y}{dx^2} > 0$, graph is concave upwards.

When $\frac{d^2y}{dx^2} < 0$, graph is concave downwards.

when $x < 0$, eg. $x = -1$

$\frac{d^2y}{dx^2} = \frac{1}{2}(e^{-1} + e^{1}) = 1.54$

when $x > 0$, eg. $x = 1$

$$\frac{d^2y}{dx^2} = \frac{1}{2}(e^1 + e^{-1}) = 1.54$$

as both values are positive, and no other turning or inflection points exist, **graph is always concave upwards.**

The sketch graph is shown in fig2.0a.

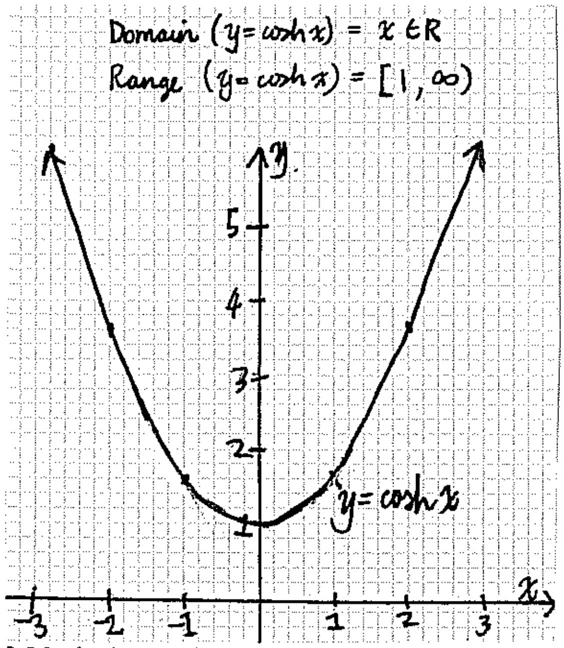



*fig2.0a sketch curve of $y = \cosh x$ using conventional methods.*

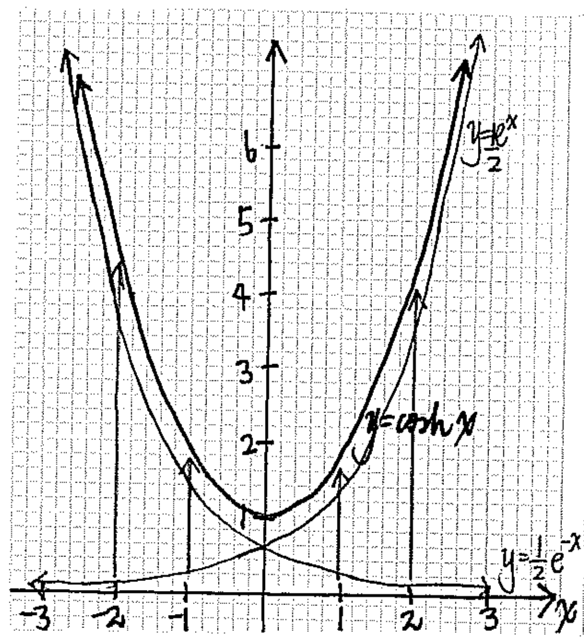



*fig2.0b sketch curve of $y = \cosh x$ using addition of ordinates method.*

The graph can be verified by the 'addition of ordinates' method of curve sketching using components of the cosh graph, ½ $e^x$ and ½ $e^{-x}$(fig2.0b). Although there didn't seem to be any horizontal asymptotes, $y = \cosh x$ actually approaches graphs of $y = ½\, e^x$ as $x \to \infty$, and $y = ½\, e^{-x}$ as $x \to -\infty$.

This occurs because as $x \to \infty$, $y = ½\, e^{-x} \to 0$. Values of $y = ½\, e^{-x}$ decrease, so $y = \cosh x$ approaches $y = ½\, e^x$:

$$\cosh x = \frac{1}{2}(e^x + e^{-x})$$

$\therefore$ as $e^{-x} \to 0$,

$$\cosh x = \frac{1}{2}e^x + 0 = \frac{1}{2}e^x$$

This reasoning can be applied to the values of $x$ as $y = \cosh x \to -\infty$.

$y = \cosh x$ is an "even" function (where $f(x) = f(-x)$) as:

$\cosh(-x) = \frac{1}{2}(e^{x} + e^{-x})$

$\therefore \cosh x = \cosh(-x)$

$\therefore$ it is symmetrical about the $y$-axis.

### SKETCH GRAPH OF SINH X:

let $y = \sinh x$

$\sinh x = \frac{1}{2}(e^{x} - e^{-x})$

$\therefore y = \frac{1}{2}(e^{x} - e^{-x})$

### X, Y INTERCEPTS

$x$-intercept at $y = 0$

$\therefore \quad 0 = \frac{1}{2}(e^{x} - e^{-x})$

$\quad 0 = e^{x} - e^{-x}$

$\therefore \quad e^{x} = e^{-x}$

$\therefore \quad x = -x$

$\quad 2x = 0$

$\therefore \quad x = 0$

$y$-intercept at $x = 0$

$y = \frac{1}{2}(e^{0} - e^{0})$

$= 0.$

**$\therefore$ *x and y*-intercepts occur at point (0,0).**

### ASYMPTOTES

$\lim_{x\to\infty} (e^{x} - e^{-x}) = \infty$

$\lim_{x\to-\infty} (e^{x} - e^{-x}) = -\infty$

**no horizontal asymptotes.** As $y$ is defined for all real $x$, there are **no vertical asymptotes.**

### TURNING POINTS

occur when $\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{1}{2}(e^{x} + e^{-x})$

$0 = \frac{1}{2}(e^{x} + e^{-x})$

$= e^x + e^{-x}$ but impossible, $\therefore$ **no turning points.**

## CONCAVITY/POINTS OF INFLECTION

Inflection points occur when $\frac{d^2y}{dx^2} = 0$

$\frac{d^2y}{dx^2} = \frac{1}{2}(e^x - e^{-x})$

and $\frac{d^2y}{dx^2} = 0$

$\therefore\ 0 = \frac{1}{2}(e^x - e^{-x})$

$\therefore\ 0 = e^x - e^{-x}$

$e^x = e^{-x}$

$\therefore x = -x$

$2x = 0$

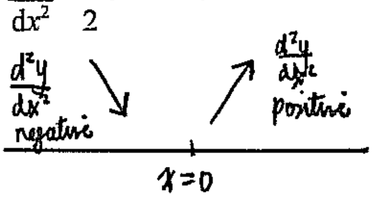
$\therefore x = 0$

When $x > 0$, eg. $x = 1$

$\frac{d^2y}{dx^2} = \frac{1}{2}(e^1 - e^{-1}) = 1.18$

When $x < 0$, eg. $x = -1$

$\frac{d^2y}{dx^2} = \frac{1}{2}(e^{-1} - e^1) = -1.18$




$\therefore$ when $x = 0$,

$y = \frac{1}{2}(e^0 - e^0) = 0$

$\therefore$ inflection point occurs at (0,0).

Graph is concave downwards when $x$ is negative and concave upwards when $x$ is positive. The sketch graph is shown in fig2.1a.

This graph can be verified by the 'addition of ordinates' method of curve sketching using components of the sinh graph, ½ $e^x$ and - ½ $e^{-x}$(fig2.1b). Although there didn't seem to be any horizontal asymptotes, $y = \sinh x$ approaches the graphs of $y = \frac{1}{2}e^x$ as $x \to \infty$, and $y = -\frac{1}{2}e^{-x}$ as $x \to -\infty$ for reasons similar to those noted on page 4.

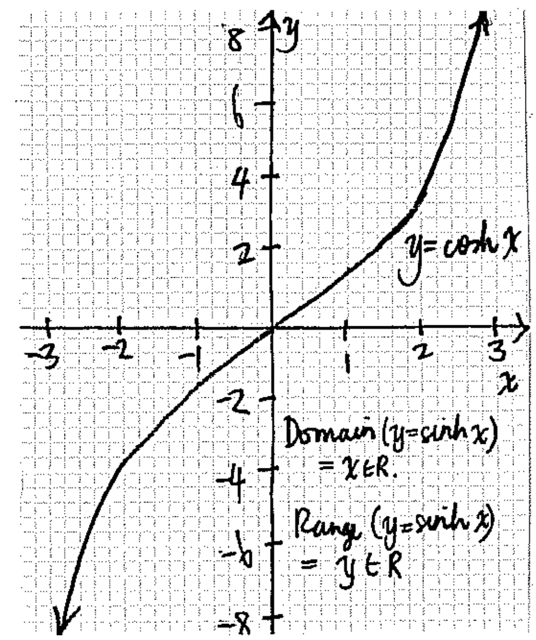



*fig2.1a sketch curve of y = sinh x using conventional methods.*

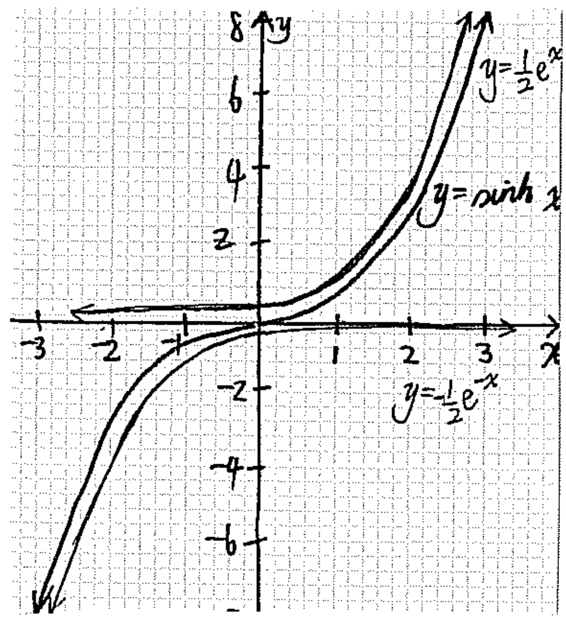



*fig2.1b sketch curve of y = sinh x using addition of ordinates method.*

$y = \sinh x$ is an "odd" function (where $f(-x) = -f(x)$) as:

$$\sinh(-x) = \frac{1}{2}(e^{-x} - e^{x}) = -\frac{1}{2}(e^{x} - e^{-x})$$

$\therefore \sinh(-x) = -\sinh(x)$

$\therefore$ it is symmetrical across the origin.

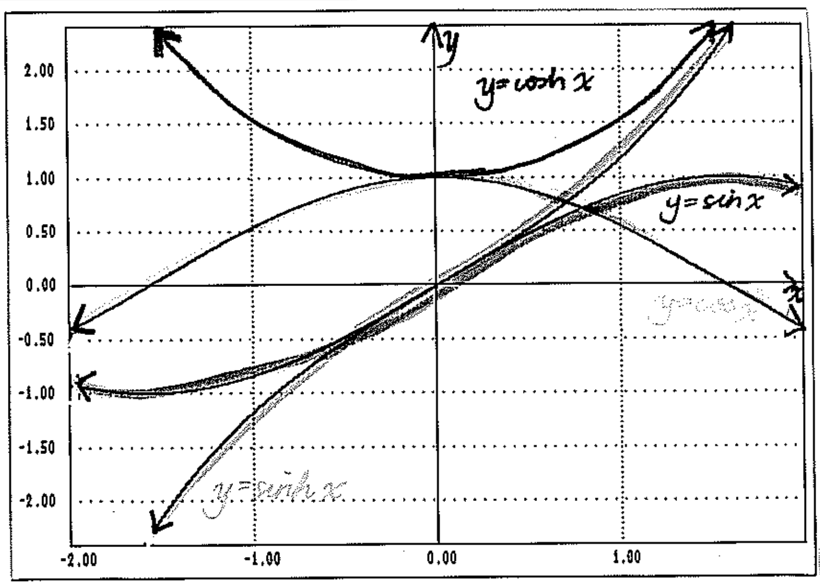



*fig2.2*

Graphs of $y = \sinh x$ and $y = \cosh x$ have been graphed on the same axis $y = \sin x$ and $y = \cos x$ (fig2.2). Within the domain of $-\pi/2 < x < \pi/2$, $y = \sinh x$ is the negative of $y = \sin x$ and $y = \cosh x$ is the inverse of $y = \cos x$. There appears to be a possible relationship between the hyperbolic and circular function graphs but that is not investigated here.

## QUESTION 2: COMPARISON OF HYPERBOLIC IDENTITIES WITH SIMILAR TRIGONOMETRIC IDENTITIES.

a) Prove $\cosh^2 x - \sinh^2 x = 1$

LHS $= \cosh^2 x - \sinh^2 x$

$= \frac{1}{4}(e^x + e^{-x})^2 - \frac{1}{4}(e^x - e^{-x})^2$

$= \frac{1}{4}[(e^{2x} + 2 + e^{-2x}) - (e^{2x} - 2 + e^{-2x})]$

$= \frac{1}{4}(e^{2x} + 2 + e^{-2x} - e^{2x} + 2 - e^{-2x})$

$= \frac{4}{4}$

$= 1$

$\therefore$ LHS = RHS

b) Prove $\cosh(2x) = 1 + 2\sinh^2 x$

RHS $1 + 2\sinh^2 x \quad = 1 + 2[\frac{1}{2}(e^{2x} - 2 + e^{-2x})]$[1]

$= 1 + \frac{1}{2}e^{2x} - 1 + \frac{1}{2}e^{-2x}$

$= \frac{1}{2}(e^{2x} + e^{-2x})$

$= \cosh(2x)$

$\therefore$ RHS = LHS

c) Show $(\cosh x + \sinh x)^n \quad = \cosh nx + \sinh nx$

RHS $= \cosh nx + \sinh nx$

$= \frac{1}{2}(e^{nx} + e^{-nx}) + \frac{1}{2}(e^{nx} - e^{-nx})$

$= \frac{1}{2}[e^{nx} + e^{-nx} + e^{nx} - e^{-nx}]$

$= \frac{1}{2}(2e^{nx})$

$= (e^x)^n$

LHS $= (\cosh x + \sinh x)^n$

$= [\frac{1}{2}(e^x + e^{-x}) + \frac{1}{2}(e^x - e^{-x})]^n$

$= [\frac{1}{2}(e^x + e^{-x} + e^x - e^{-x})]^n$

$= [\frac{1}{2}(2e^x)]^n$

[1] See appendix 1

$= (e^x)^n$

$\therefore$ RHS = LHS

d) Show that $\frac{d}{dx}(\cosh x) = \sinh x$

LHS $\frac{d}{dx}(\cosh x) = \frac{d}{dx}\left(\frac{e^x + e^{-x}}{2}\right)$

$= \frac{e^x - e^x}{2}$

$= \sinh x$

$\therefore$ LHS = RHS

These hyperbolic identities are compared to similar trigonometric identities in table3.1

| Hyperbolic identity | Similar trigonometric identity |
|---|---|
| 1) $\cosh^2 x - \sinh^2 x = 1$ | $\sin^2 x + \cos^2 x = 1$ |
| 2) $\cosh(2x) = 1 + 2\sinh^2 x$ | $\cos(2x) = 1 - 2\sin^2 x$ |
| 3) $(\cosh x + \sinh x)^n = \cosh nx + \sinh nx$ | DeMoivre's theorem which states: $(r \text{ cis } x)^n$ or $(r\cos x + r\,i\sin x)^n$ where $i = \sqrt{-1}$ $= r^n \cos nx + r^n\, i \sin nx$ |
| 4) $\frac{d}{dx}(\cosh x) = \sinh x$ | $\frac{d}{dx}(\cos x) = -\sin x$ |

*.table3.1- hyperbolic and trigonometric identities.*

There appears to be similar identities within the circular and hyperbolic functions, however, $-[(\sinh x)^2]$ corresponds to $(\sin x)^2$ whilst $\cosh x$ corresponds to $\cos x$.

$\therefore$ three other hyperbolic functions would be:

i) $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$

(by substituting $-\sinh x$ for $\sin x$ in the trigonometric identity $\cos(x+y) = \cos x \cos y - \sin x \sin y$)

RHS $\cosh x \cosh y + \sinh x \sinh y = \frac{1}{2}(e^x + e^{-x}) \times \frac{1}{2}(e^y + e^{-y}) + \frac{1}{2}(e^x - e^{-x}) \times \frac{1}{2}(e^y - e^{-y})$

$= \frac{1}{4}(e^{x+y} + e^{x-y} + e^{y-x} + e^{-x-y}) + \frac{1}{4}(e^{x+y} - e^{x-y} - e^{y-x} + e^{-x-y})$

$= \frac{1}{4}(2e^{x+y} + 2e^{-x-y})$

$= \frac{1}{2}(e^{x+y} + e^{-(x+y)})$

$= \cosh(x+y)$

= LHS

ii) $\frac{d}{dx}(\sinh x) = \cosh x$

(from the trigonometric identity $\frac{d}{dx}(\sin x) = \cos x$)

LHS $\frac{d}{dx}(\sinh x) = \frac{d}{dx}\left[\frac{1}{2}(e^x - e^{-x})\right]$

$= \frac{1}{2}(e^x + e^{-x})$

$= \cosh x$

$=$ RHS

iii) $\sinh(x-y) = \sinh y \cosh x - \cosh y \sinh x$
(from the trigonometric identity $\sin(x-y) = \sin x \cos y - \cos x \sin y$)

RHS $\cosh x \sinh y - \cosh y \sinh x = \frac{1}{2}(e^y - e^{-y}) \times \frac{1}{2}(e^x + e^{-x}) - \frac{1}{2}(e^x - e^{-x}) \times \frac{1}{2}(e^y + e^{-y})$

$= \frac{1}{4}(e^{y+x} + e^{y-x} - e^{x-y} - e^{-x-y}) - \frac{1}{4}(e^{x+y} - e^{y-x} + e^{x-y} - e^{-y-x})$

$= \frac{1}{4}(2e^{x-y} - 2e^{y-x})$

$= \frac{1}{2}(e^{x-y} - e^{-(x-y)})$

$= \sinh(x-y)$

$=$ LHS

# QUESTION 3- INVERSE HYPERBOLIC FUNCTIONS

a) $4 \sinh x = 3$

$\therefore 4[\frac{1}{2}(e^x - e^{-x})] = 3$

multiply both sides by $e^x$,

$\therefore 2e^x(e^x - e^{-x}) = 3e^x$

$2e^{2x} - 2 = 3e^x$

$\therefore 2e^{2x} - 3e^x - 2 = 0$

let $e^x = A$

$\therefore 2e^{2x} - 3e^x - 2 = 2A^2 - 3A - 2$

$\therefore 2A^2 - 3A - 2 = 0$

Using the general quadratic solution:

$A = \frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}$ where a = 2, b = -3 and c = -2

then $A = \frac{3 \pm \sqrt{[9 - 4(2)(-2)]}}{4}$

$= \frac{3 \pm \sqrt{25}}{4}$

$e^x = \frac{3 \pm 5}{4}$

$\therefore x = \log_e \left(\frac{3 \pm 5}{4}\right)$

but $\frac{3 \pm 5}{4} > 0$ because the logarithm can only be taken of a positive number

$\therefore x = \log_e \frac{3 + 5}{4}$

$= \log_e 2$

b) Sinh has an inverse function over domain of real numbers as it's a one-to-one function. Cosh doesn't because it is many-to-one so its inverse is a one-to-many function. This is demonstrated using the horizontal line test (given $y$-value must have only one $x$-value). The graph of $y = \sinh x$ (fig4.1 on the next page) passes this test, whilst the graph of $y = \cosh x$ (fig4.2) fails.
For $y = \cosh x$ to have inverse function, domain of $y = \cosh x$ must be restricted to $[0, \infty)$.

c) Let $\sinh^{-1}$ be inverse of the sinh function. The domain of a function is the range of its inverse, and range of a function is the domain of its inverse, the domain and range of $y = \sinh^{-1} x$ are:
Domain ($y = \sinh^{-1} x$) = $(-\infty, \infty)$; Range ($y = \sinh^{-1} x$) = $(-\infty, \infty)$

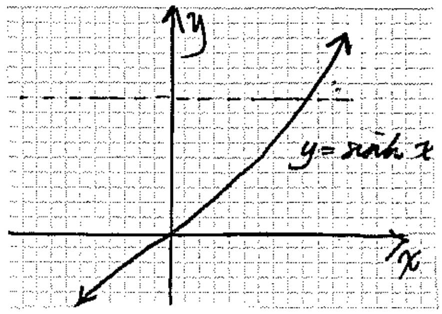



*fig4.1-horizontal line test for y = sinh x*

Components of sinh $x$ are exponential functions whose inverse functions are logarithms, therefore

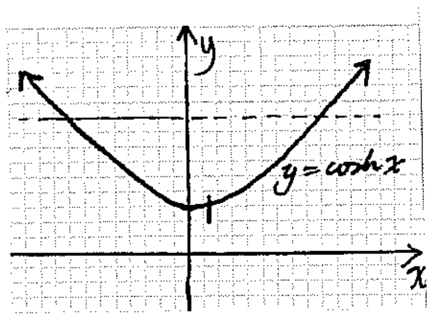



*fig4.2-horizontal line test for y = cosh x*

$\sinh^{-1} x$ can be expressed as a logarithm:

let $\sinh^{-1} x = y$

$\therefore x = \sinh y$

$\therefore x = \frac{1}{2}(e^{y} - e^{-y})$

multiply both sides by $e^{y}$,

$\therefore xe^{y} = \frac{1}{2} e^{y}(e^{y} - e^{-y})$

$2xe^{y} = e^{2y} - 1$

$\therefore 0 = e^{2y} - 2xe^{y} - 1$

let $e^{y} = A$

$\therefore\ 0 = e^{2y} - 2xe^{y} - 1$

becomes $0 = A^{2} - 2xA - 1$

using the general quadratic solution,

$A = \frac{-b \pm\sqrt{(b^{2} - 4ac)}}{2a}$ where $a = 1$, $b = -2x$ and $c = -1$

$A = \frac{2x \pm\sqrt{[4x^{2} - 4(1)(-1)]}}{2}$

$= \frac{2x \pm\sqrt{(4x^{2} + 4)}}{2}$

$= \dfrac{2x \pm \sqrt{[4(x^2+1)]}}{2}$

$= \dfrac{2x \pm 2\sqrt{(x^2+1)}}{2}$

$= x \pm \sqrt{(x^2+1)}$

but since $A = e^y$,

$e^y \quad = x \pm \sqrt{(x^2+1)}$

$\therefore y \quad = \log_e (x \pm \sqrt{(x^2+1)})$

but $x \pm \sqrt{(x^2+1)} > 0$, because only positive logarithms can be taken,

$\therefore \sinh^{-1} x = \log_e (x + \sqrt{(x^2+1)})$ where $x \in R$ ----(1)

Check Question 3a:

$4 \sinh x = 3$

$\sinh x = \dfrac{3}{4}$

$x = \sinh^{-1} \dfrac{3}{4}$

substitute this into ----(1)

$\sinh^{-1} \dfrac{3}{4} \quad = \log_e \left(\dfrac{3}{4} + \left[\dfrac{3}{4}^2 + 1\right]\right)$

$= \log_e \left(\dfrac{3}{4} + \left[\dfrac{9}{16} + 1\right]\right)$

$= \log_e \left(\dfrac{3}{4} + \dfrac{9+16}{16}\right)$

$= \log_e \left(\dfrac{3}{4} + \dfrac{1}{4}\sqrt{25}\right)$

$= \log_e 2$

## QUESTION 4- VECTORS AND THEIR RELATIONS TO HYPERBOLIC FUNCTIONS.

Verify that if the point P has position vector **r** = cos θ **i** + sin θ **j**, where θ is any real number, then it is on the unit circle.

Assuming:
- θ is not defined as an angle
- P has position vector **r** = cos θ **i** + sin θ **j**
- **i** is unit vector in $x$ direction
- **j** is unit vector in $y$ direction

∴ parametric equations are:

$x = \cos\theta$

$y = \sin\theta$

but $\sin\theta = \pm\sqrt{(1-\cos^2\theta)}$

$\therefore y = \pm\sqrt{(1-x^2)}$

$\therefore y^2 + x^2 = 1$

This is the equation for a unit circle, hence P is on the unit circle (fig5.1).

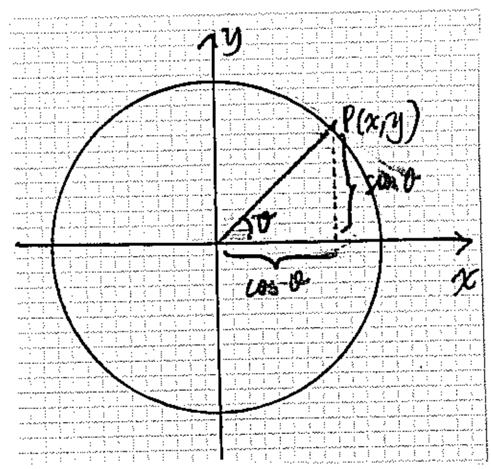


*fig5.1 the unit circle*

By definition, in a unit circle, as cosθ is $x$ coordinate and sinθ is $y$ coordinate, any point on the unit circle can be represented by cosθ **i** + sinθ **j**.

The cartesian equation of the locus of a point Q, with position vector **r** = cosh t **i** + sinh t **j**, where t ε R can be found:

Assuming:
- **i** is unit vector in $x$ direction
- **j** is unit vector in $y$ direction

parametric equations:

$$x = \cosh t = \frac{1}{2}(e^t + e^{-t})$$

$$y = \sinh t = \frac{1}{2}(e^t - e^{-t})$$

so $x^2 = \cosh^2 t$

$y^2 = \sinh^2 t$

$$\therefore x^2 - y^2 = \cosh^2 t - \sinh^2 t$$
$$= 1$$

$\therefore$ the equation of locus is

$x^2 - y^2 = 1$

# QUESTION 5- INTERPRETATION OF THE VARIABLE T

a) Area of a circle is:

$$\pi r^2$$

$\therefore$ area of the region A (fig6.1) is:

$$\frac{\pi r^2 \theta}{2\pi}$$

$$= \frac{r^2 \theta}{2}$$

$\therefore \theta = \frac{2A}{r^2}$ but $r = 1$

$\therefore \theta = 2A$

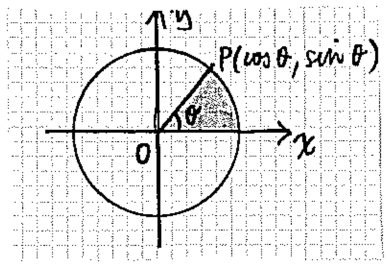



*fig6.1θ = 2A*

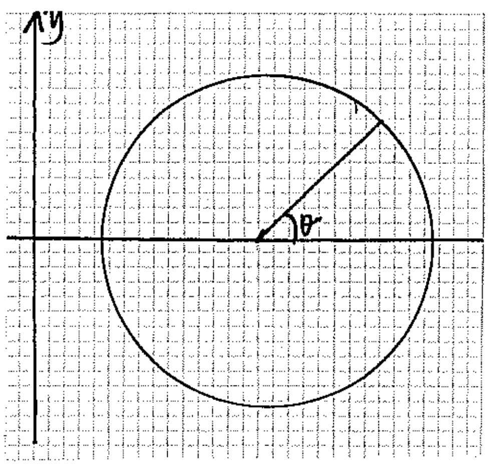



*fig6.2*

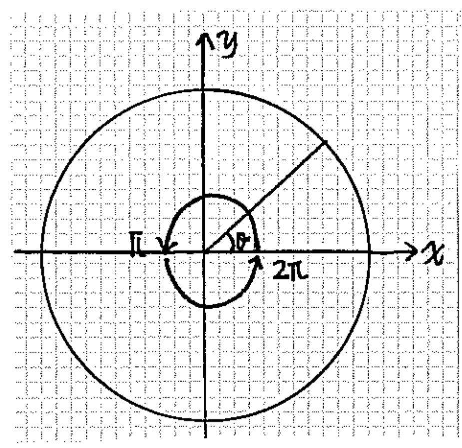



*fig6.3.-a circle assumes all angles between 0 and 2π.*

b i) The variable t in **r** = cosh t **i** + sinh t **j** cannot be similarly interpreted as an angle because:

• in a), angle θ (taken from centre of circle to perimeter and bounded by the x-axis-fig6.2) is defined as $0 < \theta < 2\pi$ ie. it is periodic. This is possible for a circle because it is a continuous function which assumes the full range of angles (fig6.3).

ie. $\mathbf{r}(\theta) = \mathbf{r}(\theta + 2\pi)$ ----(1)

This periodicity can be demonstrated by proving ---- (1).

LHS $\mathbf{r}(\theta) = \cos\theta\, \mathbf{i} + \sin\theta\, \mathbf{j}$

RHS $\mathbf{r}(\theta + 2\pi) = \cos(\theta + 2\pi)\, \mathbf{i} + \sin(\theta + 2\pi)\, \mathbf{j}$

but $\cos\theta = \cos(\theta + 2\pi)$ and $\sin\theta = \sin(\theta + 2\pi)$, $\therefore$ LHS = RHS

• the angle t in the hyperbola is taken from its focus to the curve and bounded by the x-axis (fig6.4). If t was an angle, it would be restricted and not assume the full range.

It is also non-periodic as shown below using same proof for the periodicity of the angles in the circle:
assume **r** (t) = **r** (t + 2π)
LHS **r** (t) = cosh t **i** + sinh t **j**

RHS cosh (t +2π) **i** + sinh (t +2π) **j**
but cosh (t+2π) = $(e^{t+2\pi} + e^{-t+2\pi})$ ---(2)
and cosh t = $(e^{t} + e^{-t})$ ---(3)

Equation (2) ≠ (3), hence t is not periodic, so cannot be an angle.

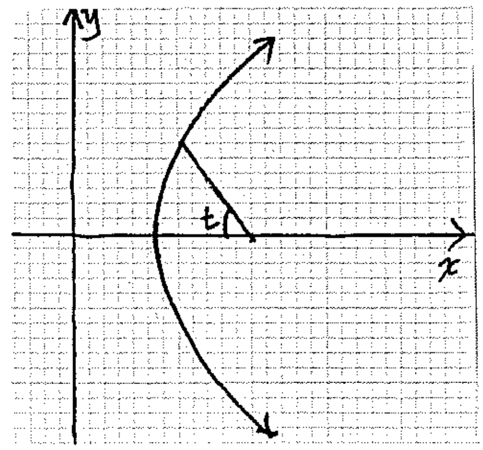


*fig6.4*

ii) Show that t = 2B, B ≥ 0 because it's the magnitude of the shaded area. Hence, t ≥ 0.

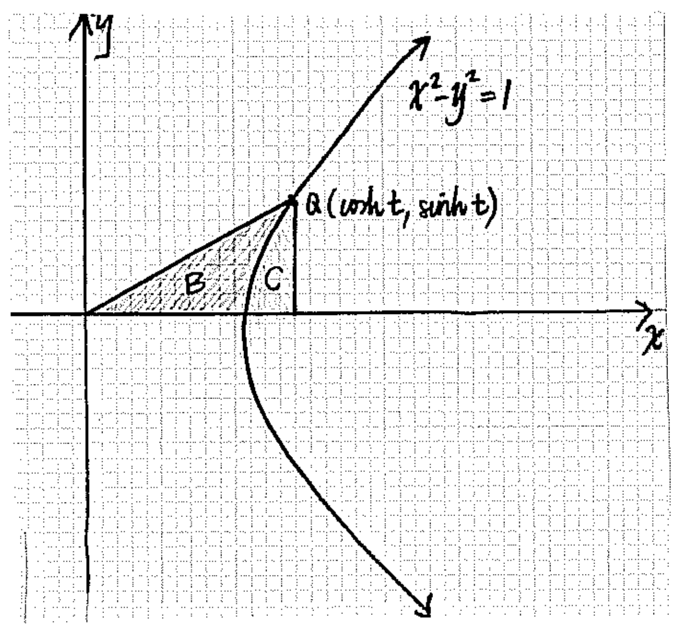


*fig7.1*

From fig 7.1, Area B = Area (C + B) - Area C

**Area C**
$x^2 - y^2 = 1$
$\therefore y = \pm\sqrt{(x^2 - 1)}$

but C is bounded by the curve, the $x$-axis and $x = \cosh t$, hence $y = \sqrt{(x^2 - 1)}$ should be used.

When $y = 0$,
$x^2 - 0 = 1$
$\therefore x = \pm 1$

by inspection, $x = 1$.
hence, limits of integration are $x = 1$ & $x = \cosh t$.

$\therefore C = \int_1^{\cosh t} \sqrt{(x^2 - 1)}\, dx$

let $x = \cosh u$ ---(1); $\frac{du}{dx} = \sinh u$

by change of variable rule:
$\int f(x)\, dx = \int f(u) \frac{dx}{du} du$
so $C = \int_1^{\cosh t} \sqrt{(\cosh^2 u - 1)} \frac{dx}{du}\, du$

becomes:
$C = \int_0^t \sqrt{(\cosh^2 u - 1)} \sinh u\, du$ from the hyperbolic identity, $\cosh^2 u - 1 = \sinh^2 u$

$\therefore C = \int_0^t \sqrt{(\sinh^2 u)} \sinh u\, du$

$= \int_0^t \sinh u \sinh u\, du$

$= \int_0^t \sinh^2 u\, du$ $\cosh 2u = 1 + 2\sinh^2 u$ $\therefore \sinh^2 u = \frac{1}{2}(\cosh 2u - 1)$

$= \frac{1}{2} \int_0^t \cosh^2 u - 1\, du$

$= \frac{1}{2} [\frac{1}{2} \sinh 2u - u]_0^t$

$= \frac{1}{2} (\frac{1}{2} \sinh 2t - t)$

$= \frac{1}{4} \sinh 2t - \frac{1}{2} t$

**Area (B+C)**

$= (\cosh t)(\sinh t) \frac{1}{2}$

$= \frac{1}{2} [\frac{1}{4} (e^{2t} - e^{-2t})]$

$= \frac{1}{8} (e^{2t} - e^{-2t})$

$= \frac{1}{4} \sinh 2t$

**Area B= Area (B+C) - Area C**

$= \frac{1}{4} \sinh 2t - (\frac{1}{4} \sinh 2t - \frac{1}{2}t)$

$= \frac{1}{2}t$

$\therefore t = 2B$

# QUESTION 6- ALTERNATE PROOF FOR T - 2B.

a) Show that shaded region USTV in fig8.1 has area t.

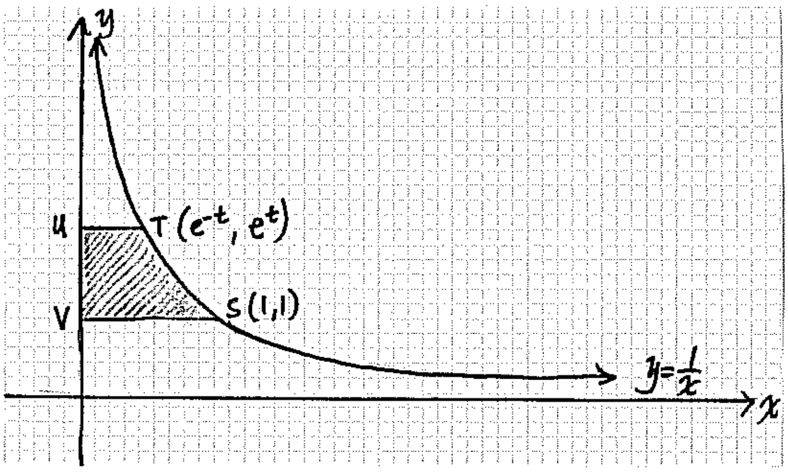



*fig8.1*

let the shaded region USTV = A, bounded by the curve $y = \frac{1}{x}$, $y = e^t$, $y = 1$ and $y$-axis

then $A = \int_1^{e^t} x\,dy$ because A is between a curve and the $y$-axis, but $y = \frac{1}{x}$ $\therefore x = \frac{1}{y}$

$$\therefore A = \int_1^{e^t} \frac{1}{y}\,dy$$

$$= [\log_e y]_1^{e^t}$$

$$= \log_e e^t - \log_e 1$$

$$= t$$

b)

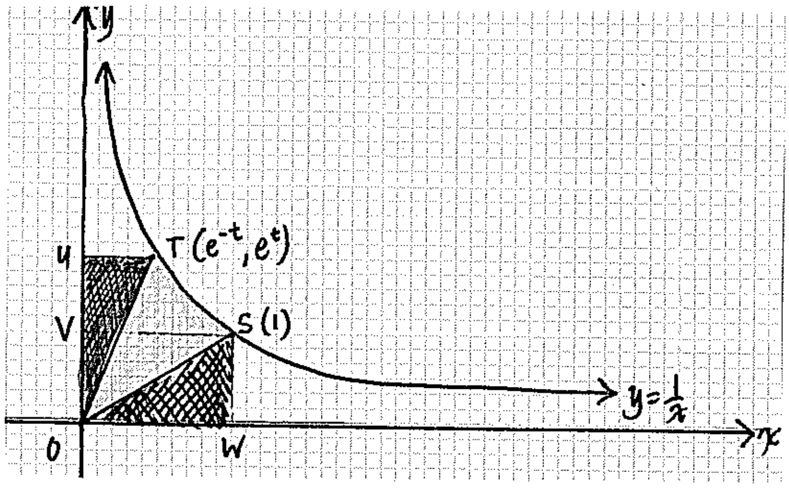



*fig9.1*

Show that green-shaded region OST (fig9.1) has area t.

Let green-shaded region OST $= A$

red-shaded region OUT $= B$

$= \frac{1}{2} \times e^{t} \times e^{-t}$

$= \frac{1}{2}$

blue-shaded region OSW $= C$

$= \frac{1}{2} \times 1 \times 1$

$= \frac{1}{2}$

area UVST $= D$

$= t$ (from Question 6a)

area VSWO $= E$

$= 1 \times 1$

$= 1$

Area A = area (D+E) - area (B+C)

$= (t + 1) - (\frac{1}{2} + \frac{1}{2})$

$= t + 1 - 1$
$= t$

c)

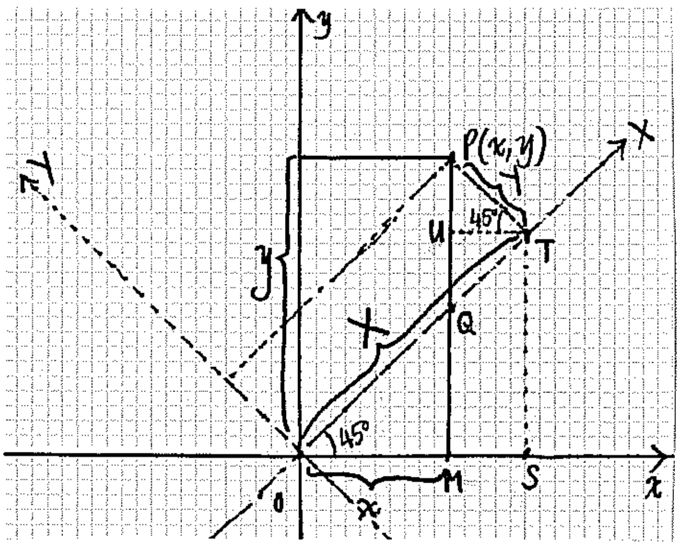



*fig10.1*

New X and Y axes inclined at 45° to the original *x*-*y* axes introduced (fig10.1). Show that point P (*x*,*y*) relative to the original axes becomes (X,Y) relative to the new axes where:

$$\mathbf{X} = \frac{1}{\sqrt{2}}(x + y)$$

$$\mathbf{Y} = \frac{1}{\sqrt{2}}(x - y)$$

This problem was dealt with in two parts. Part 1 approached from the x-axis point-of-view. Part 2 approached from y-axis point-of-view. The results were incorporated.

PART 1- from fig10.1,

$x \quad = \frac{1}{\sqrt{2}} \text{OQ}$ ------(1)

and OQ= OT - QT

but OT = X

$\therefore$OQ = X - QT --------(2)

However, $\frac{\text{UT}}{\text{QT}} = \cos 45°$ (by similar triangles) $= \frac{1}{\sqrt{2}}$

$\therefore \qquad \text{QT} = \sqrt{2}\,\text{UT}$ -----(3)

but, $\frac{UT}{PT} = \cos 45°$ (by similar triangles) $= \frac{1}{\sqrt{2}}$

and PT = Y

$\therefore \frac{UT}{Y} = \frac{1}{\sqrt{2}}$

$\therefore UT = \frac{Y}{\sqrt{2}}$

Substitute above into -----(3)

hence, $QT = \sqrt{2}\,\frac{Y}{\sqrt{2}}$

$= Y$

Substitute above into -----(2)

giving, OQ = X - Y

Substitute above into ------(1)

$\therefore x = \frac{1}{\sqrt{2}}(X - Y)$ ----(4)

make X subject,

$\therefore \frac{X}{\sqrt{2}} = x + \frac{Y}{\sqrt{2}}$

$\therefore X = \sqrt{2}\,x + Y$ -----(5)

making Y subject,

$\therefore \frac{Y}{\sqrt{2}} = \frac{X}{\sqrt{2}} - x$

$\therefore Y = X - \sqrt{2}\,x$ -------(6)

PART 2- from fig10.1,

y = NP = NU + UP

but $UP = \frac{Y}{\sqrt{2}}$ (by similar triangles)

and $NU = ST = \frac{X}{\sqrt{2}}$

$\therefore y = \frac{X}{\sqrt{2}} + \frac{Y}{\sqrt{2}}$ --------(7)

Substitute (5) into (7)

$$\therefore y = \frac{Y}{\sqrt{2}} + \frac{1}{\sqrt{2}}(\sqrt{2}\,x + Y)$$

$$= \frac{2Y}{\sqrt{2}} + x$$

$$= \sqrt{2}\,Y + x$$

making Y subject,

$$\therefore \mathbf{Y = \frac{1}{\sqrt{2}}(y - x)} \text{ ----(8)}$$

Subtituting (6) into (7)

$$\therefore y = \frac{X}{\sqrt{2}} + \frac{1}{\sqrt{2}}(X - \sqrt{2}\,x)$$

$$= \frac{2X}{\sqrt{2}} - x$$

$$= \sqrt{2}\,X - x$$

making X subject,

$$\therefore \mathbf{X = \frac{1}{\sqrt{2}}(y + x)} \text{ ----(9)}$$

equations are identical those given.

d i) From Question 6c, use equations (4) and (8).

Equation (4):

$$x = \frac{1}{\sqrt{2}}(X - Y)$$

and equation (8):

$$y = \frac{X}{\sqrt{2}} + \frac{Y}{\sqrt{2}}$$

Substitute above 2 equations into $y = \frac{1}{x}$

$$\therefore \frac{X}{\sqrt{2}} + \frac{Y}{\sqrt{2}} = \frac{1}{\frac{1}{\sqrt{2}}(X - Y)}$$

$$\therefore \left[\frac{1}{\sqrt{2}}(X + Y)\right]\left[\frac{1}{\sqrt{2}}(X - Y)\right] = 1$$

$\therefore \frac{1}{2}(X^2 - Y^2) = 1$

$\therefore X^2 - Y^2 = 2$ (equation of hyperbola relative to X-Y axes).

ii) Using equations (8) and (9) from Question 6c:

$Y = \frac{1}{\sqrt{2}}(y - x)$

$X = \frac{1}{\sqrt{2}}(y + x)$

S-coordinate on $x$-$y$ axes is (1,1). On X-Y axes:

$X = \frac{1}{\sqrt{2}}(1 + 1) = \frac{2}{\sqrt{2}} = \sqrt{2}$

$Y = \frac{1}{\sqrt{2}}(1 - 1) = 0$

$\therefore$ S-coordinate on X-Y axes is $(\sqrt{2}, 0)$.

T-coordinate on $x$-$y$ axes is $(e^{-t}, e^{t})$. On X-Y axes:

$X = \frac{1}{\sqrt{2}}(e^{-t} + e^{t})$

multiply by $\frac{\sqrt{2}}{\sqrt{2}}$ (rationalize denominator)

$\therefore X = \frac{2}{\sqrt{2}}(e^{-t} + e^{t})$

$= \sqrt{2} \cosh t$

$Y = \frac{1}{\sqrt{2}}(e^{t} - e^{-t})$

$= \frac{2}{\sqrt{2}}(e^{t} - e^{-t})$

$= \sqrt{2} \sinh t$

$\therefore$ T-coordinate on XY axes is $(\sqrt{2} \cosh t, \sqrt{2} \sinh t)$. See fig11.1

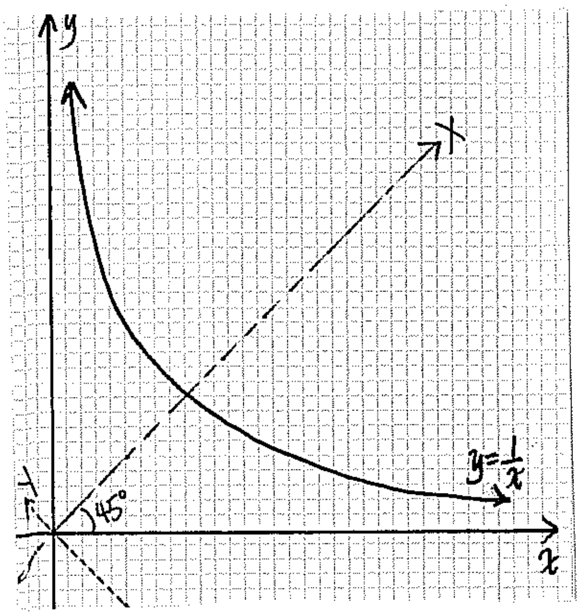



*fig11.1a graph of y = 1/x on the original x-y axes.*

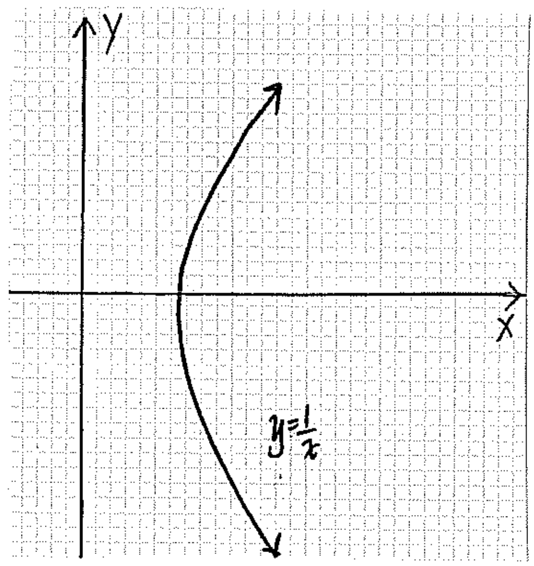



*fig11.1b graph of y = 1/x on the new X-Y axes.*

Fig11.1b shows this graph resembling that of $x^2 - y^2 = 1$ in Question 5b.

Comparing graphs of $X^2 - Y^2 = 2$ and $x^2 - y^2 = 1$ on same set of axes (fig11.2) shows $X^2 - Y^2 = 2$ to be a dilated version of $x^2 - y^2 = 1$.

Upon further inspection, this is true. The relative $x$ and $y$ coordinates on $x^2 - y^2 = 1$ have been multiplied by a factor of $\sqrt{2}$ to obtain corresponding coordinates on $X^2 - Y^2 = 2$.

eg. The $x$-intercept (1,0) on $x^2 - y^2 = 1$ curve, when dilated by a factor of $\sqrt{2}$, gives coordinate ($\sqrt{2}$,0), corresponding to $x$-intercept on $X^2 - Y^2 = 2$ curve.

Hence, the $X^2 - Y^2 = 2$ graph is the $x^2 - y^2 = 1$ graph dilated by a factor of $\sqrt{2}$.

As dilations multiply lengths of segments by the magnitude of a factor $k$, the area enclosed by the segment will subsequently increase by the magnitude of that factor squared ie.area will increase by factor of $k^2$.[2]

This reasoning is applied to evaluate the green-shaded area OST in $X^2 - Y^2 = 2$ curve (fig11.2). OST corresponds to the blue-shaded area with magnitude B in the graph of $x^2 - y^2 = 1$. Because $X^2 - Y^2 = 2$ is a dilation of $x^2 - y^2 = 1$ by a factor of $\sqrt{2}$,

[2] Appendix 2

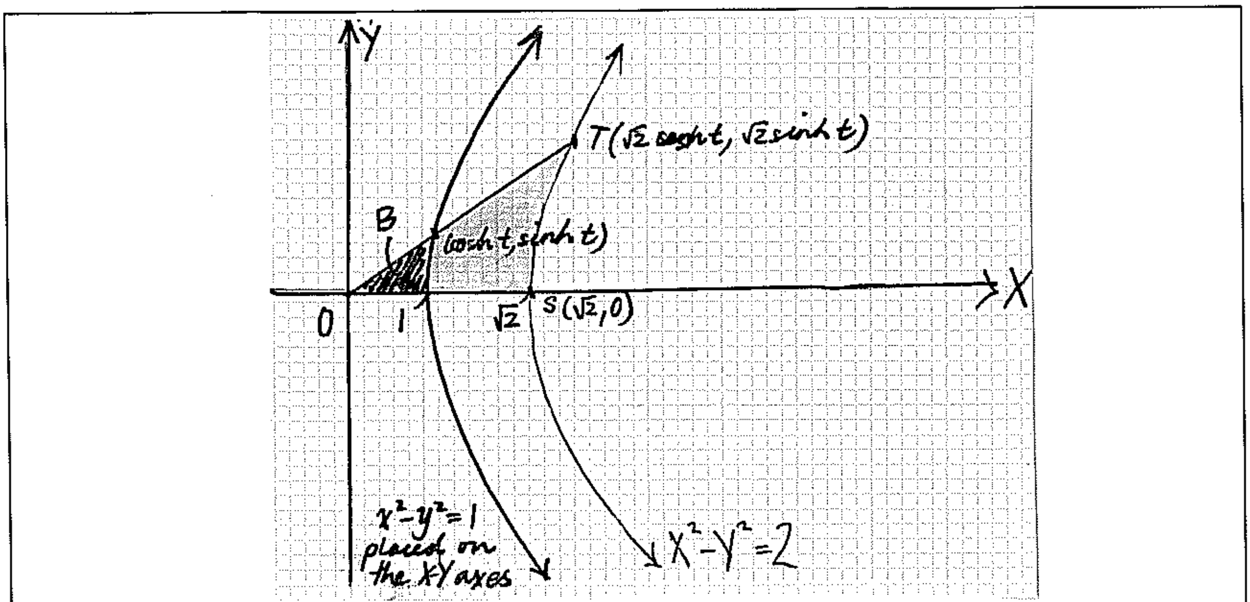



*fig11.2*

Area OST = B × $(\sqrt{2})^2$

$\therefore$ Area OST = 2B

in Question 6b, area OST = t

$\therefore$ t = 2B

Dilation factor of $\sqrt{2}$ is directly related to the function $X^2 - Y^2 = 2$. Consider rectangle in fig11.3, placed on an X-Y axes where

length $a$ = X and

length $b$ = Y

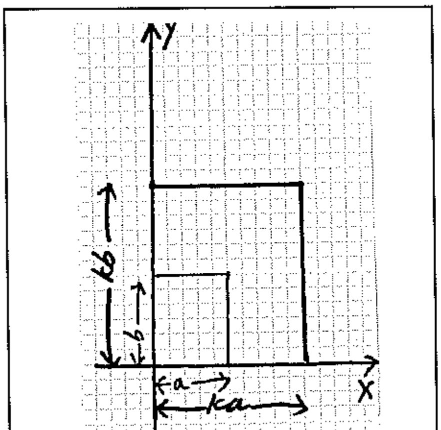



*fig11.3*

The dilation factor $k$ is applied, $\therefore$ magnitude of distance $x$ from the origin (0,0) increases by $k$.

$\therefore$ dilation factor affects distance $x$ from origin.

This is true for circular and hyperbolic functions:

- Circular functions- dilation factor affects radius of the circle (the distance on the curve furthest away from the origin; also the $x$-intercept ). Take circle with equation:

$x^2 + y^2 = 1$ ---- (1)

$x$-intercept occurs when $y = 0$ ie.at point (1,0)

When dilated by a factor of 2, $x$ value becomes:

$1 \times 2 = 2$

$y$ value:
$0 \times 2 = 0$

Hence, the new $x$-intercept after dilation occurs at (2,0)- fig11.4. This is also the radius, which means the radius has doubled.

Substituting (2,0) into -----(1): $2^2 + 0^2 = 4$

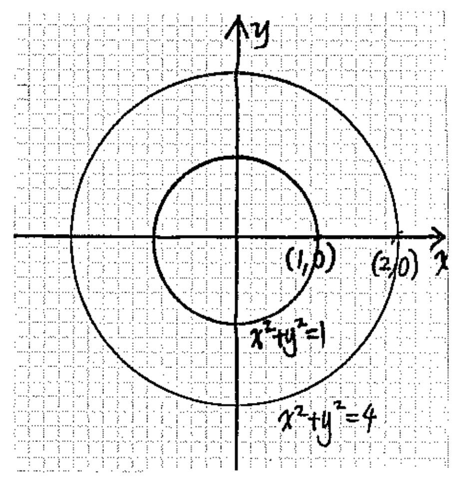


*fig11.4*

$\therefore$ circular equation after dilation is:
$x^2 + y^2 = 4$

which can be expressed as $x^2 + y^2 = c$

In this form, $c$ is the square of the dilation factor.

This same reasoning can be applied to the hyperbolic function:

$x^2 - y^2 = 1$ ----(2) ,can be expressed in the form
$x^2 - y^2 = c$.

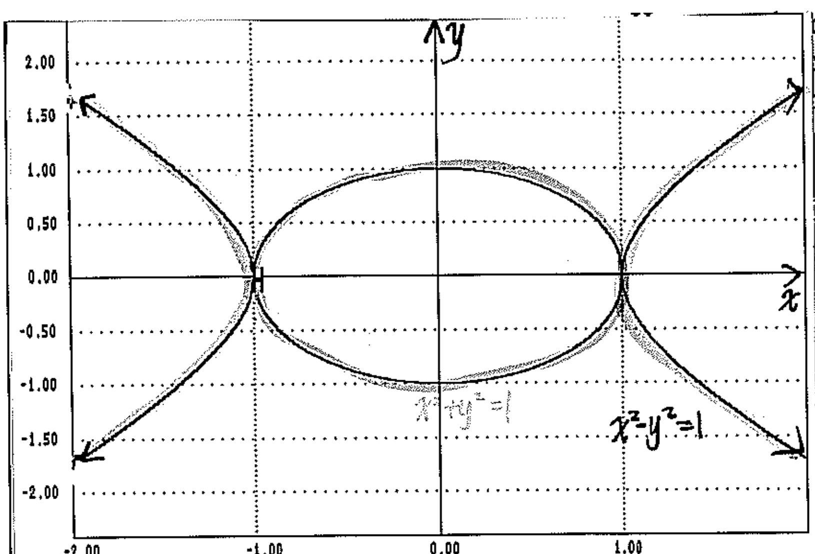


*fig11.5*

difference is:
e $x$ is the point on equation (1)
rigin, in equation (2), distance $x$
sest to origin (fig11.5).

$X^2 - Y^2 = 2$ is the dilation of
factor of √2.

## ACKNOWLEDGEMENTS


- my teacher for correcting the draft and looking at my work
- my friends for their support and understanding, for not getting impatient with me and my 'stress attacks'
- Geraldine for helping me cut words.


## APPENDICES

**Appendix 1- Working out $\sinh^2 x$**

$$\sinh^2 x = \left[\frac{1}{2}(e^x - e^{-x})\right]^2$$

$$= \frac{1}{4}(e^{2x} - 2 + e^{-2x})$$

**Appendix 2- Dilation**

Proof that when lengths of segments are multiplied by the magnitude of a factor $k$, the area enclosed by the segment will subsequently increase by the magnitude of that factor squared ie. the area will increase by a factor of $k^2$.

This can be illustrated by using the rectangle with side length $a$ and width $b$ (fig11.3).
The area of this rectangle (A) is:

$$A = \text{length} \times \text{width}$$
$$= ab$$

When $a$ and $b$ are multiplied by a factor $k$ ie. dilated, the new length becomes $ka$ and the new width becomes $kb$.
However, the new area ($A_1$) is:

$$A_1 = \text{length} \times \text{width}$$
$$= ka \times kb$$
$$= k^2 ab$$

Hence, the area has increased by a factor of $k^2$.