SPECIALST MATHEMATICS

Common Assessment Task 1: Problem-solving task

Problem 2 – Estimating definite integrals.

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INTRODUCTON

The aim of this investigation is to explore various methods for calculating the area bounded by a function, and determining which method gives the greatest accuracy. Techniques that will be investigated include;

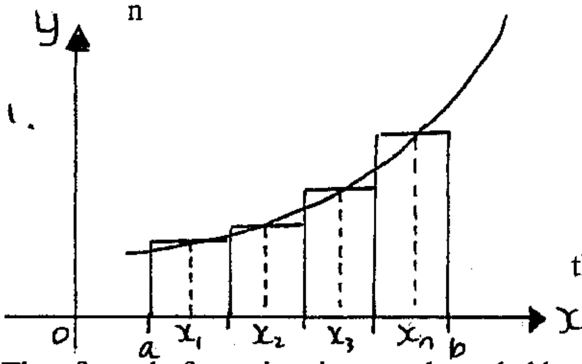
1. Definite integrals.
2. The midpoint rule.
3. The trapezoidal rule.
4. Gauss' point rules.

QUESTION 1

a. Firstly we shall investigate approximating definite integrals using midpoint rule. A numerical method of approximating region bounded by curve of x-axis and the ordinates, $x = a$ and $x = b$ is to calculate sum of the areas of rectangles. To calculate width of subdivisions, use the formula,

$w = \dfrac{b - a}{n}$. (see diagram 1)

DIAGRAM 1.



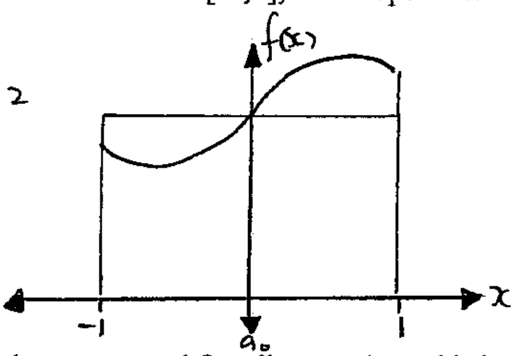
Let $x_1, x_2, x_3 \ldots x_n$ = x-coordinates of the midpoint of the subdivisions.

Thus formula for estimating area bounded by graph using midpoint rule is given by,

$$\boxed{\int_{-1}^{1} f(x)\,dx \approx w\left[f(x_1) + f(x_2) + f(x_3) + \ldots\ldots f(x_n)\right]}$$

a. Using midpoint rule with one interval gives exact answer when used to estimate $\int f(x)\,dx$, with $f(x) = a_1x + a_0$, where a_1 and a_0 are any real valued constants. Within the interval [-1,1], the midpoint will be $2a_0$. (diagram 2)

DIAGRAM 2



$$\therefore \int_{-1}^{1} a_1x + a_0 dx = \left[\frac{a_1x^2}{2} + a_0x\right]_{-1}^{1}$$

$$= \left(\frac{a_1}{2} + a_0\right) - \left(\frac{a_1}{2} - a_0\right)$$

$$= 2a_0$$

As these are equal for all a x and a , this is proof the midpoint rule with one interval, gives exact value for $\int_{-1}^{1} f(x)\,dx$

b. We now prove that if midpoint rule (with one interval) was used to estimate higher degree polynomial, namely quadratic, result can never equal exact value of $\int_{-1}^{1} f(x)$ dx.

$$f(x) = a_2x^2 + a_1x + a_0\ ,\ a_2 \neq 0$$

$$\text{Thus} = \int_{-1}^{1} a_2x^2 + a_1x + a_0\,dx$$

$$= \left[\frac{a_2x^3}{3} + \frac{a_1x^2}{2} + a_0x\right]_{-1}^{1}$$

$$= \left(\frac{a_2}{3} + \frac{a_1}{2} + a_0\right) - \left(-\frac{a_2}{3} + \frac{a_1}{2} - a_0\right)$$

$$= \frac{2a_2}{3} + 2a_0$$

As a ≠0, midpoint evaluation will not equal exact value.

QUESTION 2

a. Now investigate a rule which gives exact estimate when $\int_{-1}^{1} f(x)dx$ is quadratic polynomial. We are given rule stating

$$\int_{-1}^{1} f(x)dx \approx w[f(-v) + f(v)]$$

From previous answer, $\int_{-1}^{1} a_2x^2 + a_1x + a_0 dx = \frac{2a_2}{3} + 2a_0$

$\therefore \quad \frac{2a_2}{3} + 2a_0 \approx w[f(-v) + f(v)]$ Now by substitution,

$$f(-v) = a_2v - a_1v + a_0$$
$$f(v) = a_2v + a_1v + a_0$$

$$\frac{2a_2}{3} + 2a_0 \approx w[(a_2v - a_1v + a_0) + (a_2v + a_1v + a_0)]$$

$$\frac{2a_2}{3} + 2a_0 \approx w[2a_2v^2 + 2a_0]$$

$$\frac{2a_2}{3} + 2a_0 \approx 2w[a_2v^2 + a_0]$$

$$\frac{2a_2}{3} + 2a_0 \approx 2wa_2v^2 + 2wa_0$$

$\frac{2a_2}{3} = 2wa_2v^2$ and $2a_0 = 2wa_0$

(equation 1) (equation 2)

By solving equation 2. $2 = 2w$

$$\boxed{\therefore w = 1}$$

substitute value into equation 1

$$\frac{2a_2}{3} = 2wa_2v^2$$

$$\therefore \frac{2a_2}{3} = 2(1)a_2v^2$$

$$\therefore \frac{2a_2}{3} = 2a_2v^2$$

$$\therefore \frac{2}{3} = 2v^2$$

$$\therefore v^2 = \frac{1}{3}$$

$\therefore v = +\dfrac{1}{\sqrt{3}}$ restriction in which $v > 0$

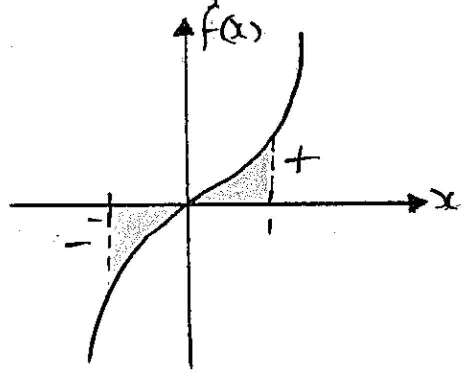
$\therefore v = \dfrac{1}{\sqrt{3}}$

Gauss **2-point rule**. We can refer to this rule as $\int_{-1}^{1} f(x)dx \approx f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$

b. When using Gauss 2-point rule to estimate a cubic polynomial, it will also give the same exact answer. The reason for this is additional term of a_3x^3, is known as an 'odd' function of x.

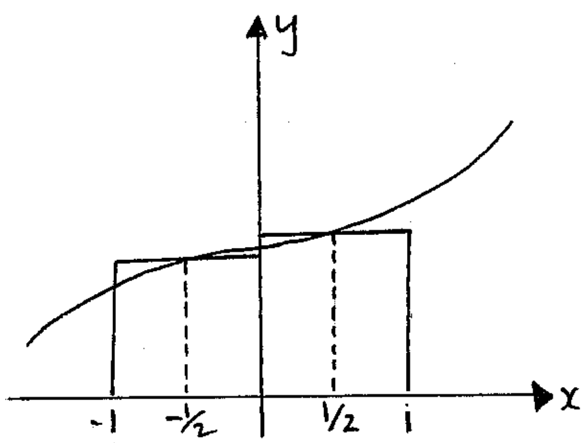
So $\int_{-1}^{1} a_3x^3\,dx = 0$

Proof $\left[\dfrac{a_3x^4}{4}\right]_{-1}^{1} = \dfrac{a_3}{4} - \dfrac{a_3}{4} = 0$



DEFINITION: An odd function, when $f(-x) = f(x)$, is one which is symmetrical about the origin. For instance, if we take the graph of $y = x^3$ and integrate it between [-1,1], both areas will cancel each other out.

c. If midpoint rule was used to estimate $\int_{-1}^{1} f(x)\,dx$ using two intervals, points at which $f(x)$ would be evaluated will be $-\frac{1}{2}$ and $\frac{1}{2}$. (diagram 4)



$$\int_{-1}^{1} f(x)\,dx = \frac{2a_2}{3} + 2a_0$$

$$\int_{-1}^{1} f(x) \approx w\left[f\left(-\tfrac{1}{2}\right) + f\left(\tfrac{1}{2}\right)\right]$$

$$f\left(\frac{-1}{2}\right) = \frac{1a_2}{4} - \frac{1a_1}{2} + a_0$$

$$f\left(\frac{1}{2}\right) = \frac{1a_2}{4} + \frac{1a_1}{2} + a_0$$

$$f\left(\frac{-1}{2}\right) + f\left(\frac{1}{2}\right) = \frac{1a_2}{2} + 2a_0$$

Even though midpoint rule does use two subdivisions, it will not give exact answer as in (a).

QUESTION 3

Now use calculus to find definite integrals of different functions. Then use Gauss 2-point rule to find exact approximation, hence calculate percentage error when using 2-point rule.

a. (I) $\int_{-1}^{1} (x+1)\sqrt{x^2+2x+2}\,dx$

let $u = x^2 + 2x + 2$

$\dfrac{du}{dx} = 2x + 2$

$\therefore \dfrac{1}{2}\dfrac{du}{dx} = x + 1$

When $x = 1$, $u = (1)^2 + 2(1) + 2$
$= 5$

$x = -1$, $u = (-1)^2 + 2(-1) + 2$
$= 1$

$$\int_{-1}^{1} (x+1)\sqrt{x^2+2x+2}\,dx = \int_{-1}^{1} \frac{1}{2}\sqrt{u}\,\frac{du}{dx}.dx$$

$$= \frac{1}{2}\int_{-1}^{1} \sqrt{u}\,du$$

$$= \frac{1}{2}\int_{-1}^{1} u^{\frac{1}{2}}\,du$$

$$= \frac{1}{2}\times\frac{2}{3}\left[\,u^{\frac{3}{2}}\right]_{-1}^{1}$$

$$= \frac{1}{3}(5) - \frac{1}{3}(1)$$

$$= 3.72677 - 0.33333$$

$$\boxed{3.393}$$

(ii) $\int_{-1}^{1} \dfrac{2}{4+x^2}\,dx = \int \dfrac{2}{2^2+x^2}\,dx$

$$= \operatorname{Tan}^{-1}\left(\frac{x}{2}\right)$$

$$= \operatorname{Tan}^{-1}\left(\frac{1}{2}\right) - \operatorname{Tan}^{-1}\left(\frac{-1}{2}\right)$$

$$= 0.463647 - -0.436476$$

$$\boxed{0.927}$$

(iii) $\int_{-1}^{1} \sin^2 3x \, dx$

Double angle formula.

$$\cos 2x = 1 - 2\sin^2 x$$

$$\therefore \quad 1 - \cos 2x = 2\sin^2 x$$

$$\therefore \quad \frac{1}{2} - \cos 2x = \sin^2 x$$

$$\therefore \quad \frac{1}{2} - \cos 6x = \sin^2 3x$$

So $\int_{-1}^{1} \sin^2 3x \, dx = \int_{-1}^{1} (1 - \cos 6x) \, dx$

$$= \left[\frac{x}{2} - \frac{1}{12}\sin^2 6x\right]_{-1}^{1}$$

$$= \left(\frac{1}{2} - \frac{1}{12}\sin^2 6\right) - \left(-\frac{1}{2} - \frac{1}{12}\sin -6\right)$$

$$= 0.523284 - -0.523284$$

$$\boxed{1.047}$$

We will estimate each of the above integrals using the Gauss 2-point rule.

b. $\int_{-1}^{1} (x+1)\sqrt{x^2 + 2x + 2} \, dx$

Gauss 2-point rule states $\int_{-1}^{1} f(x) \, dx \approx f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$

Therefore we substitute values of $\frac{-1}{\sqrt{3}}$ and $\frac{1}{\sqrt{3}}$ into $f(x)$

$$f\left(\frac{1}{\sqrt{3}}\right) = \left(1 + \frac{1}{\sqrt{3}}\right)\sqrt{\left(\frac{-1}{\sqrt{3}}\right)^2 + 2\,\frac{-1}{\sqrt{3}} + 2}$$

$$= \left(1 + \frac{1}{\sqrt{3}}\right)\sqrt{\frac{1}{3} + \frac{2}{\sqrt{3}} + 2}$$

$$= 2.94590$$

$$f\left(\frac{-1}{\sqrt{3}}\right) = \left(1 - \frac{1}{\sqrt{3}}\right)\sqrt{\left(\frac{-1}{\sqrt{3}}\right)^2 + 2\,\frac{-1}{\sqrt{3}} + 2}$$

$$= \left(1 - \frac{1}{\sqrt{3}}\right)\sqrt{\frac{1}{3} + \frac{2}{\sqrt{3}} + 2}$$

$$= 0.458849$$

$$f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right) =$$

(ii) $\displaystyle\int_{-1}^{1} \frac{2}{4+x^2}dx \approx f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$

Substituting values $\frac{-1}{\sqrt{3}}$ and $\frac{1}{\sqrt{3}}$ into $f(x)$

$$f\left(\frac{-1}{\sqrt{3}}\right) = \frac{2}{4+\left(\frac{-1}{\sqrt{3}}\right)^2} = \frac{2}{4+\frac{1}{3}} = \frac{6}{13}$$

$$f\left(\frac{1}{\sqrt{3}}\right) = \frac{2}{4+\left(\frac{1}{\sqrt{3}}\right)^2} = \frac{2}{4+\frac{1}{3}} = \frac{6}{13}$$

$$\therefore f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right) = \frac{6}{13} + \frac{6}{13} = \frac{12}{13} = 0.923$$

(iii) $\displaystyle\int_{-1}^{1} \sin^2 3x \, dx \approx f\left(\frac{-1}{\sqrt{3}}\right) f\left(\frac{1}{\sqrt{3}}\right)$

Substituting values $\frac{-1}{\sqrt{3}}$ and $\frac{1}{\sqrt{3}}$ into $f(x)$.

$$f\left(\frac{-1}{\sqrt{3}}\right) = \sin^2\left(3 \times \frac{-1}{\sqrt{3}}\right)$$
$$= \sin^2(-\sqrt{3})$$

$$f\left(\frac{1}{\sqrt{3}}\right) = \sin^2\left(3 \times \frac{1}{\sqrt{3}}\right)$$
$$= \sin^2(\sqrt{3})$$

$$f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right) = \sin^2(-\sqrt{3}) + \sin^2(\sqrt{3})$$

$$= 0.97411 + 0.97422$$
$$= 1.948$$

Now determining how accurate Gauss 2-point rule is in comparison to definite integral for each finding, using following equation,

$$\boxed{\text{Percentage error} = \frac{\text{(estimate)} - \text{(exact answer)}}{\text{Exact answer}} \times 100\%}$$

c. (I) $\int_{-1}^{1} (x+1)\sqrt{x^2+2x+2}\,dx \approx f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$

$$\text{Percentage error} = \frac{(\text{estimate}) - (\text{exact answer})}{\text{Exact answer}} \times 100\%$$

$$= \frac{3.405 - 3.393}{3.393} \times 100\%$$

$$\boxed{0.354\%}$$

(ii) $\int_{-1}^{1} \frac{2}{4+x^2}\,dx \approx f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$

$$\text{Percentage error} = \frac{(\text{estimate}) - (\text{exact answer})}{\text{Exact answer}} \times 100\%$$

$$= \frac{0.923 - 0.927}{0.927} \times 100\%$$

$$\boxed{-0.431\%}$$

(iii) $\int_{-1}^{1} \sin^2 3x\,dx \approx f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$

$$\text{Percentage error} = \frac{(\text{estimate}) - (\text{exact answer})}{\text{Exact answer}} \times 100\%$$

$$= \frac{1.948 - 1.047}{1.047} \times 100\%$$

$$\boxed{86.06\%}$$

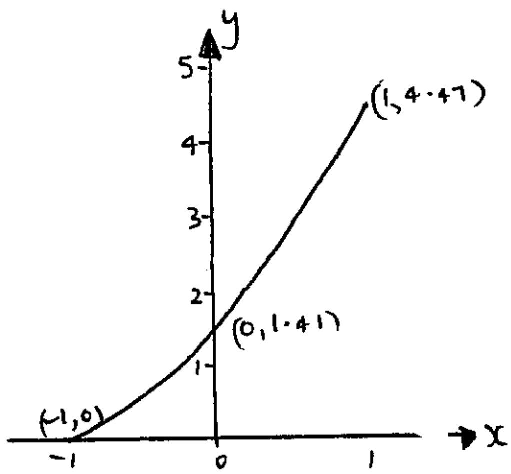
Approximating with 2-point Gauss rule gives good results for function (I) and (II). Both estimations less than 0.5% error (overestimation and underestimation respectively). However, 2-point rule performed poorly for function (III), which gave overestimation of 86.06%. This was expected, as periodic function did not display similar properties to typical cubic or lesser degree polynomial.

QUESTION 4

We will now plot the graphs of $y = f(x)$ over the interval [-1,1] for each integrated in question 3.

(I)

$y = (x+1)\sqrt{x^2+2x+2}\, dx$

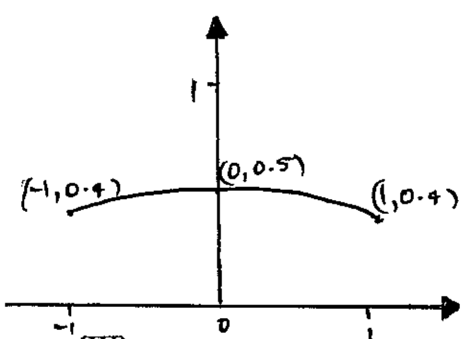


Graphs (I) and (II) represent similar paths of quadratic or cubic functions. Graph (III) does not display these features.

Graph (I) is an increasing function with positive increasing gradient. This feature is part of a quadratic or cubic function.

(II)

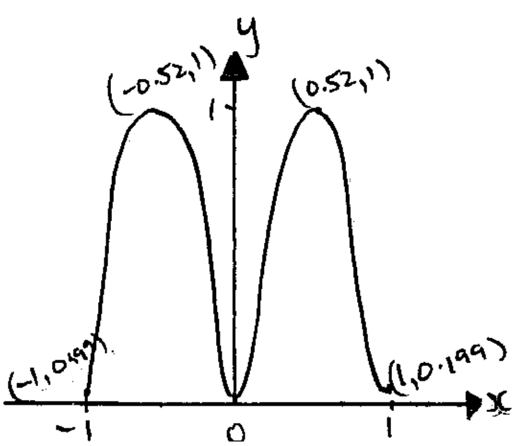
$Y = \frac{2}{4+x^2}$



Graph (II) is symmetrical about $x = 0$ and has a turning point at $x = 0$. This is typical of a particular quadratic in the form $y = -ax^2 + \frac{1}{2}$

(III)

$Y = \sin^2 3x$



Graph (III) has three turning points which occur well within interval. This feature is not typical of quadratic or cubic functions

QUESTION 5

Now going to compare Gauss 2-point rule with midpoint rule and trapezoidal rule, using same points to ensure appropriate comparison between various methods. We will also calculate percentage error for each method of approximation.

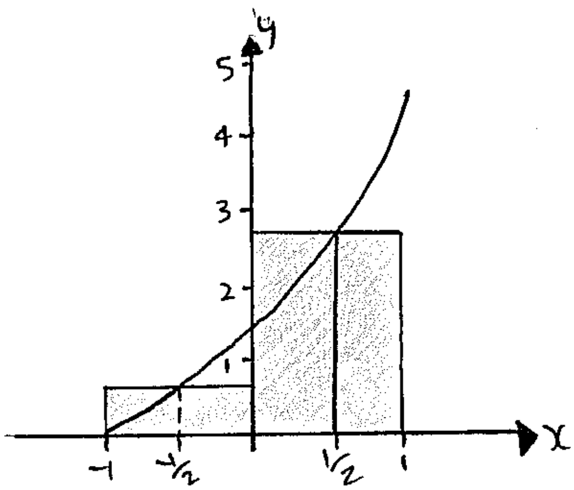
First we consider midpoint rule with **two** intervals.

a. To use midpoint rule with two intervals, use formula form before,

$$\boxed{\int_{-1}^{1} f(x)dx \approx w\left[f\left(\frac{-1}{2}\right) + f\left(\frac{1}{2}\right)\right]}$$

(w = 1 from before)

$$\int_{-1}^{1} (x+1)\sqrt{x^2+2x+2}\,dx$$



$$\therefore \int_{-1}^{1} f(x)\,dx \approx f\left(\frac{-1}{2}\right) + f\left(\frac{1}{2}\right)$$

$$f\left(\frac{-1}{2}\right) = \left(\frac{-1}{2}+1\right)\sqrt{\left(\frac{-1}{2}\right)^2 + 2\,\frac{-1}{2} + 2}$$

$$= \left(\frac{1}{2}\right)\sqrt{\frac{1}{4} + 1 + 2}$$

$$= 0.55901$$

$$f\left(\frac{1}{2}\right) = \left(\frac{1}{2}+1\right)\sqrt{\left(\frac{1}{2}\right)^2 + 2\,\frac{1}{2} + 2}$$

$$= \left(\frac{3}{2}\right)\sqrt{\frac{1}{4} + 1 + 2}$$

$$= 2.7041$$

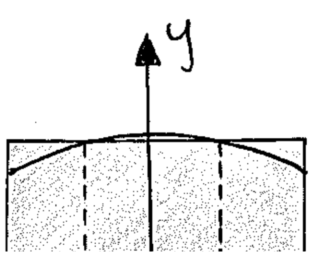
$$f\left(\frac{-1}{2}\right) + f\left(\frac{1}{2}\right) = 3.263$$

$$\text{Percentage error} = \frac{(\text{estimate}) - (\text{exact answer})}{\text{Exact answer}} \times 100\%$$

$$= \frac{3.263 - 3.393}{3.393} \times 100\%$$

$$\boxed{-3.83\%}$$

(II) $\displaystyle\int_{-1}^{1} \frac{2}{4+x^2}\,dx$



$$\int_{-1}^{1} f(x)dx \approx f\left(\frac{-1}{2}\right) + f\left(\frac{1}{2}\right) \qquad (w = 1)$$

$$f\left(\frac{-1}{2}\right) = \frac{2}{4+\left(\frac{-1}{2}\right)^2} = \frac{2}{4+\frac{1}{4}} = \frac{8}{17}$$

$$f\left(\frac{1}{2}\right) = \frac{2}{4+\frac{1}{\,}} = \frac{2}{4+\frac{1}{4}} = \frac{8}{17}$$

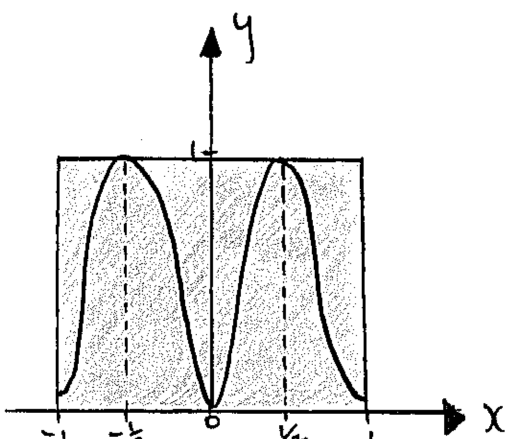
$$f\left(\frac{-1}{2}\right) + f\left(\frac{1}{2}\right) = \frac{8}{17} + \frac{8}{17} = \frac{16}{17} = 0.94118$$

$$\text{Percentage error} = \frac{(\text{estimate}) - (\text{exact answer})}{\text{Exact answer}} \times 100\%$$

$$= \frac{0.94118 - 0.927}{0.927} \times 100\%$$

$$\boxed{1.53\%}$$

(III) $\int_{-1}^{1} \sin^2 3x \, dx$



$$\therefore \int_{-1}^{1} f(x)\,dx \approx f\left(\frac{-1}{2}\right) + f\left(\frac{1}{2}\right) \qquad (w = 1)$$

$$f\left(\frac{-1}{2}\right) = \sin^2\left(3 \times \frac{-1}{2}\right)$$
$$= 0.99499$$

$$f\left(\frac{1}{2}\right) = \sin^2\left(3 \times \frac{1}{2}\right)$$
$$= 0.99499$$

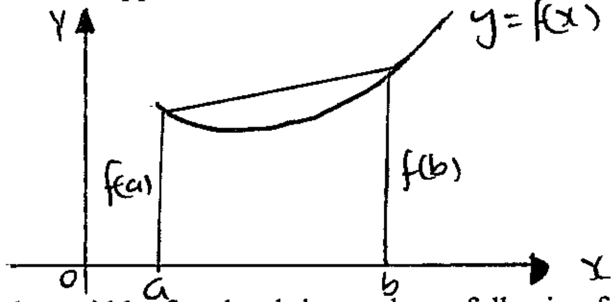
$$f\left(\frac{-1}{2}\right) + f\left(\frac{1}{2}\right) = 1.990$$

$$\text{Percentage error} = \frac{(\text{estimate}) - (\text{exact answer})}{\text{Exact answer}} \times 100\%$$

$$= \frac{1.990 - 1.047}{1.047} \times 100\%$$

$$\boxed{90.10\%}$$

Simple application of trapezoidal rule substitutes straight line in place of curve between x = a and x = b, shown in diagram below. The area of trapezium is used as an approximation to area under curve.



To calculate width of each sub-interval, use following formula,

$w = \frac{b-a}{N}$ Where N = no of sub-intervals.

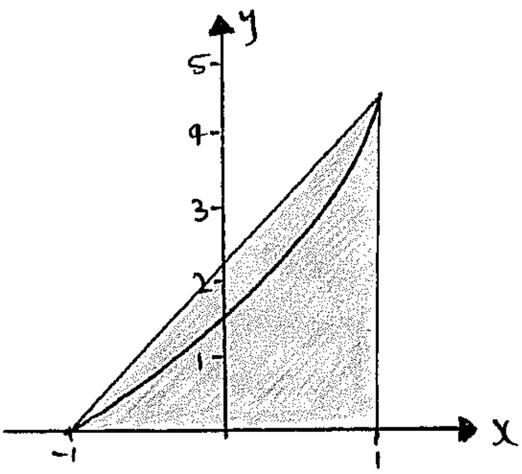
Area of trapezium is given by $\frac{1}{2}$ × the sum of the parallel sides × the distance between them, thus

$$\boxed{\int_a^b f(x)\,dx \approx \frac{1}{2}(b-a)[f(a)+f(b)]}$$

substitute value of w into our equation, general formula becomes

$$\boxed{\int_a^b f(x)\,dx \approx \frac{w}{2}[f(a)+f(b)]}$$

b. (I) $y = (x+1)\sqrt{x^2+2x+2}\,dx$ Over $[-1,1]$



$w = \frac{1--1}{1} = 2$

$$\therefore \int_{-1}^{1} f(x)\,dx \approx [f(-1) + f(1)]$$

$$\approx f(-1) + f(1)$$

$f(-1) = (-1+1)\sqrt{(-1)^2+2(-1)+2}$
$= 1$

$f(1) = (1+1)\sqrt{(1)^2+2(1)+2}$
$= 2\sqrt{5}$

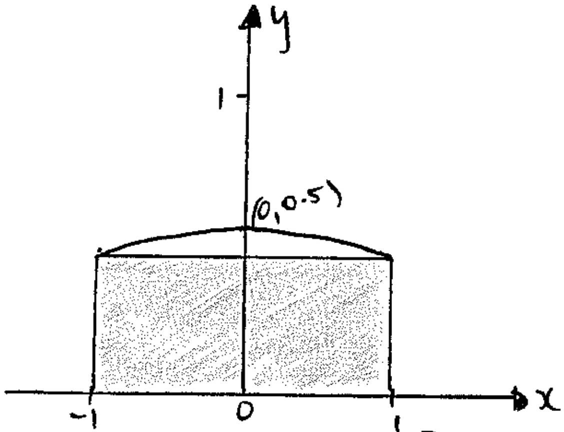
$f(1) + f(-1) = 5.472$

Percentage error $= \frac{(\text{estimate}) - (\text{exact answer})}{\text{Exact answer}} \times 100\%$

$= \frac{5.472 - 3.393}{3.393} \times 100\%$

$\boxed{61.28\%}$

(II) $y = \dfrac{2}{4+x^2}dx$



w = 2 (from previous answer)

$$\therefore \int_{-1}^{1} f(x)\,dx \approx [f(-1)+f(1)]$$

$$f(-1) = \frac{2}{4+(-1)^2} = \frac{2}{5}$$

$$f(1) = \frac{2}{4+(1)^2} = \frac{2}{5}$$

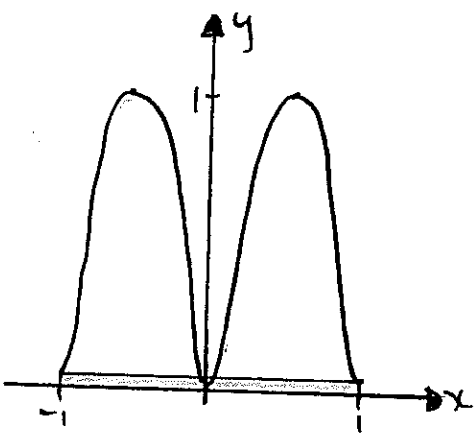
$$f(-1) + f(1) = \frac{2}{5} + \frac{2}{5} = \frac{4}{5} = 0.8$$

$$\text{Percentage error} = \frac{(\text{estimate}) - (\text{exact answer})}{\text{Exact answer}} \times 100\%$$

$$= \frac{0.8 - 0.927}{0.927} \times 100\%$$

$$\boxed{-13.7\%}$$

(III) $y = \sin^2 3x$



w = 2 (previous answer)

$$\therefore \int_{-1}^{1} f(x)dx \approx [f(-1)+f(1)]$$

$$f(-1) = \sin^2(3 \times -1)$$
$$= 0.0199$$

$$f(1) = \sin^2(3 \times 1)$$
$$= 0.0199$$

$$f(-1) + f(1) = 0.0398$$

$$\text{Percentage error} = \frac{(\text{estimate}) - (\text{exact answer})}{\text{Exact answer}} \times 100\%$$

$$= \frac{0.0398 - 1.047}{1.047} \times 100\%$$

$$\boxed{-96.2\%}$$

c.

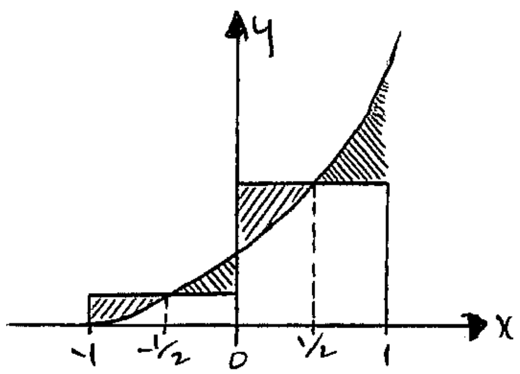
SUMMARY OF ACCURACY

	$(x+1)\sqrt{x^2+2x+2}$	$\frac{2}{4+x^2}$	$\sin^2 3x$
Midpoint	–3.83%	1.53%	90.1%
Trapezoidal	31.8%	–13.7%	–96.2%
Gauss	0.354%	–0.431%	86.06%

Clearly all three approximations lead to very poor result for $y = \sin^2 3x$. Gauss 2-point rule performed very well for (I) and (II). Midpoint rule was second best due to nature of functions. In (I) and (II), part overestimation can be counterbalanced for part underestimation.

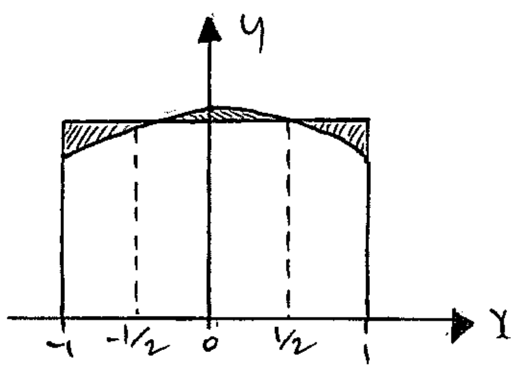
THE MIDPOINT RULE

Midpoint rule gives quite good approximations to graphs (I) and (II), but not appropriate for graph (III) as it gives very large overestimation.



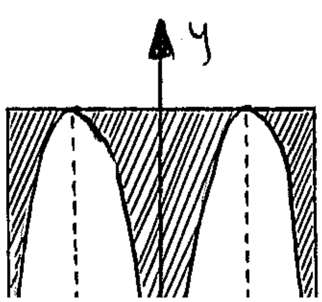
Underestimation
Overestimation

Using midpoint rule for graph (I) gives two regions where overestimation occurred, and two regions where underestimation occurred. To a large extent, these cancel each other out giving good overall approximation.



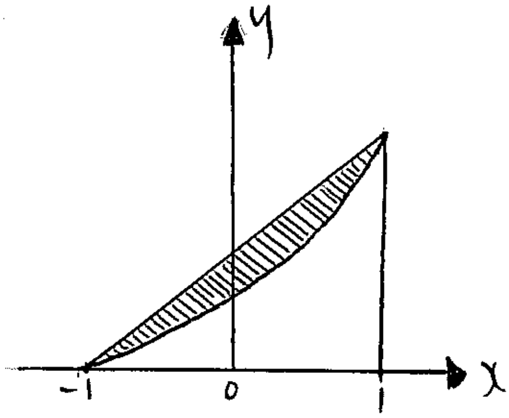
The same occurs with graph (II), however region of underestimation is a lot smaller than two regions of overestimation. Overall, midpoint rule is still quite accurate for these functions.

For graph $\sin^2 3x$, midpoint rule approximated most of area bounded by graph, however it also calculated a large portion of area surrounding graph, giving an overestimation of 90.1%.

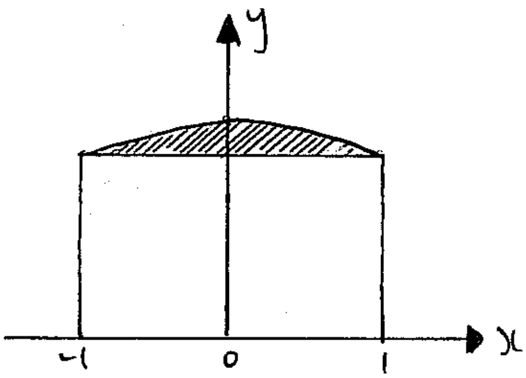


THE TRAPEZOIDAL RULE

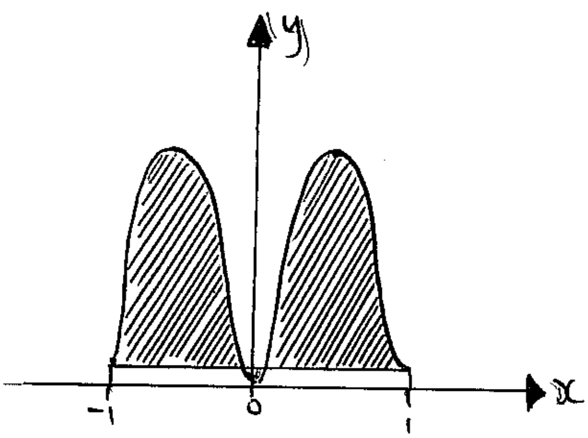
Trapezoidal rule failed to perform well with given functions. Trapezoidal rule fails to identify actual behaviour of graph, it only considers endpoints of the graph.



The best result from trapezoidal rule was from $y = \frac{2}{4+x^2}$ which gave an underestimation of 13.7%, which is still relatively large in comparison to midpoint rule.



It can be seen from illustrations how trapezoidal rule operates. Consider graph (I), gives a large overestimation due to fact that graph is concave in nature, trapezoidal rule fails to recognise this feature. Similar problem when dealing with graph (II), however this graph is convex, resulting in underestimation.



Trapezoidal rule works very inefficiently with graph (III). Both endpoints occur at bottom of graph, so trapezoidal rule will calculate this small region at bottom of graph, thus it fails to calculate the area of the full graph as most of behaviour occurs beyond this bottom point. Therefore a large underestimation occurs.

Now going to explore method for evaluating definite integral over interval $[\alpha, \beta]$, where α and β are any real numbers, while $\alpha<\beta$. We are given the substitution, $x = ku + l$, where k and l are real numbers. This will enable us to transform $\int_{\alpha}^{\beta} f(x)\,dx$ to definite integral in form $\int_{-1}^{1} g(u)\,du$.

QUESTION 6

A.

Given the substitution of $x = ku + l$ When $u = -1,\ x = \alpha$

$u = 1,\ x = \beta$

We are able to find expressions for k and l in terms of α and β, by making substitution.

$x = ku + l$ $\quad$ $x = \alpha$, when $u = -1$

$\therefore \alpha = -k + l$ $\quad$ (equation 1)

$x = ku + l$ $\quad$ $x = \beta$, when $u = 1$

$\therefore \beta = k + l$ $\quad$ (equation 2)

(equation 1) + (equation 2) $\quad$ $2l = \alpha + \beta$

$$\therefore l = \frac{\alpha + \beta}{2}$$

(equation 2) – (eqaution 1) $\quad$ $2k = \beta - \alpha$

$$\therefore k = \frac{\beta - \alpha}{2}$$

We now use these expressions to make substitution of $x = ku + l$ in the definite integral $\int_0^4 \frac{4x+4}{x^2+2x+5}$, obtaining definite integral in form $\int_{-1}^{1} g(u)\,du$. We can estimate function using 2-point rule.

$$\int_0^4 \frac{4x+4}{x^2+2x+5}\,dx$$

$\alpha = 0,$
$\beta = 4$

$$k = \frac{\beta-\alpha}{2} = \frac{4-0}{2} = 2$$

$$l = \frac{\alpha+\beta}{2} = \frac{0+4}{2} = 2$$

Thus $x = ku + l$

$x = 2u + 2$

$\therefore \underline{dx} = 2\,du$

$$\int_0^4 \frac{4(2u+2)+4}{(2u+2)+2(2u+2)+5}\,dx$$

$$\int_0^4 \frac{8u+8+4}{4u^2+8u+4+4u+4+5}\,dx$$

When $x = 0$, $u = \frac{0-2}{2} = -1$

$$\int_0^4 \frac{(8u+12)2}{4u^2+12u+13}\,dx$$

$x = 4$, $u = \frac{4-2}{2} = 1$

$$\int_{-1}^1 \frac{16u+24}{4u^2+12u+13}\,du$$

Hence we have transformed $\int_\alpha^\beta f(x)\,dx$ to a definite integral in the form $\int_{-1}^1 g(u)\,du$ by substitution. We are able to obtain an estimate $\int_{-1}^1 g(u)\,du$ using Gauss 2-point rule.

$$\int_{-1}^1 \frac{16u+24}{4u^2+12u+13}\,du \approx f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

$$f\left(\frac{-1}{\sqrt{3}}\right) = \frac{16\left(\frac{-1}{\sqrt{3}}\right) + 24}{4\left(\frac{-1}{\sqrt{3}}\right) + 12\left(\frac{-1}{\sqrt{3}}\right) + 13}$$

$$= \frac{14.76239}{7.405130} = 1.9935$$

$$f\left(\frac{1}{\sqrt{3}}\right) = \frac{16\left(\frac{1}{\sqrt{3}}\right) + 24}{4\left(\frac{1}{\sqrt{3}}\right) + 12\left(\frac{1}{\sqrt{3}}\right) + 13}$$

$$= \frac{33.2376}{21.2615} = 1.5633$$

$$f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right) = 1.5633 + 1.9935$$

$$= 3.557$$

c. Investigating $\int_0^4 \frac{4x+4}{x^2+2x+5}\,dx$ using midpoint rule (**two** intervals).

$$W = \frac{4-0}{2} = 2$$

$$\int_0^4 \frac{4x+4}{x^2+2x+5}\,dx \approx w\left[f(1) + f(3)\right]$$

$$f(1) = \frac{4(1)+4}{(1) + 2(1) + 5} = \frac{8}{8} = 1$$

$$f(3) = \frac{4(3) + 4}{(3) + 2(3) + 5} = \frac{16}{20} = 0.8$$

$$2\ \left[f(1) + f(3)\right] = 2\left[1 + 0.8\right]$$
$$= 2\left[1.8\right]$$
$$= \boxed{3.600}$$

D. Investigating $\int_0^4 \frac{4x+4}{x^2+2x+5}\,dx$ using trapezoidal rule (**one** interval)

W = 2 (from previous answer)

$$\int_0^4 \frac{4x+4}{x^2+2x+5}\,dx \approx 2\ [\ f\ (0) + f(4)]$$

$$f(0) = \frac{4(0) + 4}{(0) + 2(0) + 5} = 0.8$$

$$f(4) = \frac{4(4) + 4}{(4) + 2(4) + 5} = \frac{20}{29} = 0.689655$$

$$2\left[f(0) + f(4)\right] = 2\left[0.8 + 0.689655\right]$$
$$= \boxed{2.979}$$

e. Evaluating $\int_0^4 \frac{4x+4}{x^2+2x+5}\,dx$ using calculus.

$$\int_0^4 \frac{4x+4}{x^2+2x+5}\,dx$$

$$= 2\int \frac{2x+2}{x^2+2x+5}\,dx$$

$$= 2\left[\log_e(x^2+2x+5)\right]$$

$$= 2\left[\log_e 29 - \log_e 5\right]$$

$$= 2\log_e \frac{29}{5}$$

$$\boxed{3.516}$$

2-point rule

$$\text{Percentage error} = \frac{(\text{estimate}) - (\text{exact answer})}{\text{Exact answer}} \times 100\%$$

$$= \frac{3.557 - 3.516}{3.516} \times 100\%$$

$$\boxed{1.71\%}$$

Midpoint rule

$$\text{Percentage error} = \frac{(\text{estimate}) - (\text{exact answer})}{\text{Exact answer}} \times 100\%$$

$$= \frac{3.600 - 3.516}{3.516} \times 100\%$$

$$\boxed{2.39\%}$$

Trapezoidal rule

$$\text{Percentage error} = \frac{(\text{estimate}) - (\text{exact answer})}{\text{Exact answer}} \times 100\%$$

$$= \frac{2.979 - 3.516}{3.516} \times 100\%$$

$$\boxed{-15.27\%}$$

Results are consistent with previous results. It seems possible relative accuracy's of methods are the same over any interval. (We would need to investigate other intervals and functions if wishing to draw firm conclusions).

QUESTION 7

A. Determining rule which gives exact answer for higher degree polynomial, namely function. The rule states,

$$\int_{-1}^{1} f(x)\,dx \approx w_1\left[f(-v) + w_2 f(0) + w_1 f(v)\right]$$

This will enable us to attain exact answer when $f(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x^1 + a_0$ where a_0, a_1, a_2, a_3 and a_4 are any real valued constants.

Determining values of v , w_1, and w_2.

$$f(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$$

$$\int_{-1}^{1} a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0\,dx = \left[\frac{a_4x^5}{5} + \frac{a_3x^4}{4} + \frac{a_2x^3}{3} + \frac{a_1x^2}{2} + a_0x\right]_{-1}^{1}$$

$$= \left(\frac{a_4}{5} + \frac{a_3}{4} + \frac{a_2}{3} + \frac{a_1}{2} + a_0\right) - \left(\frac{-a_4}{5} + \frac{a_3}{4} - \frac{a_2}{3} + \frac{a_1}{2} - a_0\right)$$

$$= \boxed{\frac{2a_4}{5} + \frac{2a_2}{3} + 2a_0}$$

$$f(-v) = w_1[a_4v - a_3v + a_2v - a_1v + a_0]$$

$$f(0) = w_2[0 + 0 + 0 + 0 + a_0] = w_2[a_0]$$

$$f(v) = w_1[a_4v + a_3v + a_2v + a_1v + a_0]$$

$$\therefore \quad w_1 f(-v) + w_2 f(0) + w_1 f(v)$$

$$= w_1[a_4v^4 - a_3v^3 + a_2v^2 - a_1v + a_0] + w_2[a_0] + w_1[a_4v^4 + a_3v^3 + a_2v^2 + a_1v + a_0]$$

$$= \boxed{2a_4v^4w_1 + 2a_2v^2w_1 + 2a_0(w_1 + w_2)}$$

$$\int_{-1}^{1} f(x)\,dx \approx w_1 f(-v) + w_2 f(0) + w_1 f(v)$$

$$\therefore \frac{2a_4}{5} + \frac{2a_2}{3} + 2a_0 \approx 2a_4 v^4 w_1 + 2a_2 v^2 w_1 + 2a_0(w_1 + w_2)$$

Equating coefficients

$$2v^4 w_1 = \frac{2}{5} \qquad \text{(equation 1)}$$

$$2v^2 w_1 = \frac{2}{3} \qquad \text{(equation 2)}$$

$$2w_1 + w_2 = 2 \qquad \text{(equation 3)}$$

solve equation 1 and equation 2 simultaneously.

$$\frac{2/5}{2/3} = \frac{2v^4 w_1}{2v^2 w_1}$$

$$\frac{3}{5} = v^2$$

$$v = \pm\sqrt{\frac{3}{5}} \quad \text{However } v > 0 \quad \therefore v = \sqrt{\frac{3}{5}}$$

Substitute value in equation 2

$$2w_2 \frac{3}{5} = \frac{2}{3}$$

$$\frac{3w_2}{5} = \frac{1}{3} \qquad \therefore w_2 = \frac{5}{9}$$

Substitute value in equation 3

$$2w_1 + w_2 = 2$$

$$2 \;\frac{5}{9} + w_1 = 2$$

$$w_1 = 2 - 2\;\frac{5}{9}$$

$$w_1 = \frac{8}{9}$$

Gauss 3-point rule. $\int_{-1}^{1} f(x)\,dx \approx \frac{5}{9} f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{\frac{3}{5}}\right)$

b. The Gauss 3-point rule gives exact answer when evaluating
$f(x) = a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$, where a_0, a_1, a_2, a_3, a_4, and a_5, are any real value constants, because additional term of x is odd function.

$$\int_{-1}^{1} a_5x^5\,dx = \frac{a_5x^6}{6} = \frac{a_5}{6} - \frac{a_5}{6} = 0$$

Gauss 3-point rule gives exact answer with this function.

c. Now are going to evaluate $\int_{1}^{1.5} e^{-x^2}\,dx$ using the 2-point rule. In order to do this, we must make substitution in the form $x = ku + l$

$$\int_{-1}^{1} f(x)\,dx = \int e^{-x^2}\,dx$$

$u = 1,\ x = \alpha$
$u = 1.5,\ x = \beta$

$x = ku + l$
when $u = 1,\ x = \alpha$
$\alpha = k + l$ (equation 1)

$x = ku + l$
when $u = 1.5, x = \beta$
$\beta = 1.5k + l$ (equation 2)

Solve simultaneously in terms of l.
Equation 1 + Equation 2

$2l = \beta + \alpha$

$\therefore l = \frac{\beta + \alpha}{2} = \frac{1.5 + 1}{2} = \frac{5}{4}$

Equation 2 – Equation 1

$2k = \beta - \alpha$

$\therefore\ k = \frac{\beta - \alpha}{2} = \frac{1.5 - 1}{2} = \frac{1}{4}$

$x = ku + l$
$\therefore x = \frac{1}{4}u + \frac{5}{4}$

$$u = \frac{x - 5/4}{1/4}$$

$$u = 4x - 5$$

$$\frac{du}{dx} = 4$$

$$du = 4dx$$

when $x = 1.5$ $u = \dfrac{3/2 - 5/4}{1/4} = 1$ when $x = 1$, $u = \dfrac{1 - 5/4}{1/4} = -1$

$$\therefore \int_1^{1.5} e^{-x^2}\,dx = \int_{-1}^{1} \frac{1}{4} e^{-\frac{1}{16}(u+5)^2}\,du$$

Now apply Gauss' 2-point rule

$$\int_{-1}^{1} f(x)\,dx \approx g\left(\frac{-1}{\sqrt{3}}\right) + g\left(\frac{1}{\sqrt{3}}\right)$$

$$\int_{-1}^{1} \frac{1}{4} e^{-\frac{1}{16}(u+5)^2}\,du \approx g\left(\frac{-1}{\sqrt{3}}\right) + g\left(\frac{1}{\sqrt{3}}\right)$$

$$g\left(\frac{-1}{\sqrt{3}}\right) = \frac{1}{4} e^{-\frac{1}{16}\left(\frac{-1}{\sqrt{3}}+5\right)^2}$$

$$= 0.07362$$

$$g\left(\frac{1}{\sqrt{3}}\right) = \frac{1}{4} e^{-\frac{1}{16}\left(\frac{1}{\sqrt{3}}+5\right)^2}$$

$$= 0.03578$$

$$g\left(\frac{-1}{\sqrt{3}}\right) + g\left(\frac{1}{\sqrt{3}}\right) = 0.03578 + 0.07362$$

$$\boxed{0.1094}$$

Using the same method but applying Gauss' **3-point rule**

$$\int_{-1}^{1} f(x)dx \approx \frac{5}{9}\, g\left(\sqrt{\frac{-3}{5}}\right) + \frac{8}{9}\, g(0) + \frac{5}{9}\, g\left(\sqrt{\frac{3}{5}}\right)$$

$$\int_{-1}^{1} f(x)dx = \int_{-1}^{1} \frac{1}{4} e^{-\frac{1}{16}(u+5)^2} du$$

$$\frac{5}{9}\, g\left(\sqrt{\frac{-3}{5}}\right) = \frac{1}{4} e^{-\frac{1}{16}\left(\sqrt{\frac{3}{5}}+5\right)^2}$$

$$= 0.04551$$

$$\frac{8}{9}\, g(0) = \frac{1}{4} e^{-\frac{1}{16}(5)^2}$$

$$=0.04658$$

$$\frac{5}{9}\, g\left(\sqrt{\frac{3}{5}}\right) = \frac{1}{4} e^{-\frac{1}{16}\left(\left(\sqrt{\frac{3}{5}}\right)+5\right)^2}$$

$$= 0.01728$$

$$\therefore \frac{5}{9}\, g\left(\sqrt{\frac{-3}{5}}\right) + \frac{8}{9}\, g(0) + \frac{5}{9}\, g\left(\sqrt{\frac{3}{5}}\right) = 0.04551 + 0.04658 + 0.01728$$

$$\boxed{0.10937}$$

(iii) We will now estimate $\int_{1}^{1.5} e^{-x^2} dx$ using the midpoint rule with various numbers of intervals. For ease, a graphics calculator and Microsoft excel aided in varying the numbers of intervals. Table 1 contains a summary of results from various intervals. (see appendices for data)

Intervals	Results
50	0.110401
60	0.109139
70	0.109342
80	0.109364
85	0.109343
90	0.109788

It can be seen from the table that as you increase your intervals, the more accurate your results become. When using 2-point Gauss rule, it gave an accuracy of 0.1094, while the 3-point Gauss rule gave an accuracy of 0.1094. Therefore we can conclude form Gauss' rules and the summary of results, that in order to get a precise estimate, you would need to divide your interval up into around 85 subdivisions. This gives almost the same answer as the Gauss 2-point rule and 3-point rule.

CONCLUSION

After investigating various methods of integration, I found that there was no real substitute for definite Intograls. Even though other methods give close answers, they don't give the exact answer