Series and Application

Annuities: Example

An annuity is a fund into which money is invested regularly (normally yearly-annual, hence the name). They include superannuation funds and trust funds. Each payment will earn interest over a smaller period than the one before, and it will be seen how each amount is calculated separately:

Steve deposits $50 into a superannuation fund at the start of each month. The fund pays 15% p.a. interest which is compounded at the end of each month. Find the value of the fund at the end of 10 years (to the nearest dollar).

$$There\ is\ 120\ months, monthly\ rate = \frac{0.15}{12} = 0.0125$$

$$A_1 = 50(1.0125)^{120}$$

$$A_2 = 50(1.0125)^{119}$$

$$A_3 = 50(1.0125)^{118}$$

$$\vdots$$

$$A_{120} = 50(1.0125)$$

$$Total\ Amount = 50(1.0125 + 1.0125^2 \ldots 1.0125^{120})$$

Last part is a GS with $a = 1.0125, r = 1.0125, n = 120$

$$So\ S_{120} = \frac{1.0125(1.0125^{120} - 1)}{1.0125 - 1} = 278.62715$$

$\underline{So\ total\ amount = 50 \times 278.62715 = \$13\ 933}$

$(nearest\ \$)$

Compound Interest

Compound interest is a Geometric series with formula:

$$A = P(1 + r)^n\ where$$

$$P = principal\ (initial\ amount)$$

$$r = interest\ rate\ as\ a\ decimal$$

$$n = number\ of\ time\ periods$$

Example: Compound Interest

A Person places $100 000 into a bank account that pays 6% interest, compounded every 6 months. How long will it take for the account to be worth $1 000 000? :

$$Need\ to\ find\ n, r = \frac{0.06}{2} = 0.03$$

$$P = \$100{,}000\ and\ A = \$1{,}000{,}000$$

$$1{,}000{,}000 = 100{,}000(1.03)^n$$

$$\frac{1{,}000{,}000}{100{,}000} = (1.03)^n$$

$$10 = (1.03)^n$$

$$\log(10) = \log(1.03)^n = n \log 1.03$$

$$n = \frac{\log 10}{\log 1.03} = 77.89845726$$

$$It\ will\ take\ at\ least\ 78\ years\ for\ the\ account\ to\ be\ worth\ \$1{,}000{,}000$$

Loan Repayments: Example

The formulas for compound interest and geometric series can be used to calculate regular loan repayments:

A man borrows $7000 to buy a new car. He has to pay off the loan over 4 years in equal monthly payments. If the interest rate is 15 % p.a.and is compounded monthly, find the amount of each payment:

$$The\ inital\ amount\ is\ 7000,$$

$$No.of\ monthly\ payments = 12 \times 4 = 36. r = \frac{0.15}{12} = 0.0125$$

Amount Owing after 1 month: $A_1 = 7000(1.0125) - M$

$$A_2 = A_1(1.0125) - M$$

$$= \big((7000(1.0125) - M)(1.0125)\big) - M$$

$$= 7000(1.0125)^2 - M(1.0125) - M$$

$$= 7000(1.0125)^2 - M(1.0125 + 1)$$

$$\vdots$$

$$A_{36} = 7000(1.0125)^{36} - M(1.0125^{36-1} + \cdots + 1.0125 + 1)$$

Last part is a GS with a=1 r=1.0125

$$S_{36} = \frac{1.0125^{36} - 1}{1.0125 - 1} \approx 45.1155055$$

End value equals 0 → $0 = 7000(1.0125)^{36} - M(\frac{1.0125^{36} - 1}{1.0125 - 1})$

$$7000(1.0125)^{36} = M(\frac{1.0125^{36} - 1}{1.0125 - 1})$$

$$\frac{7000(1.0125)^{36}}{\frac{1.0125^{36} - 1}{1.0125 - 1}} = M \approx \$242.6572995\ or\ \$242.66\ (nearest\ cent)$$

Series and Application

Types of Series:

General, Arithmetic and Geometric

nth term of a series

The nth term of a series is denoted with T_n

General (no generic formula)

1+2+3+4+…

1+1+2+3+5+8+…

Arithmetic Series

Arithmetic Series

In this type of series, each term is a constant amount more than the previous term. This amount is known as the **common difference** and is assigned the letter *d*.

$$d = T_2 - T_1 = T_3 - T_2$$

nth term of an Arithmetic Series

For arithmetic series with first term *a*, common difference *d*, T_n:

$$T_n = a + (n-1)d$$

Partial sum of an Arithmetic Series

The sum of the first *n* terms (*n*th partial sum):

$$S_n = \frac{n}{2}(a + l) \text{ where } l = \text{last or nth term}$$

In general,

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

Example: Partial Sum

An AS has T_1=15 and T_2=30. Find the sum of the first 32 terms:

$T_1 = a = 15$

$T_2 = a + (n-1)d$

$30 = 15 + d$

$d = 30 - 15 = 15$

$So\ a = 15, d = 15, n = 32$

$S_{32} = \frac{32}{2}(2(15) + (31)15)$

$S_{32} = 16 \times 495 = 7{,}920$

Sigma Notation

Sigma notation uses the Greek letter, Sigma ($\sum$), which stands for the sum of a series.

For example:

$$\sum_q^p f(n) \text{ is the sum of terms } (f(n))n \text{ starting at } q, \text{and ending at } p.$$
$$Note: \text{number of terms in the series is } (p - q + 1)$$

Geometric Series

Example: Partial Sum

A GS has $T_3 = 6.4, r = \frac{1}{8}$

Find the difference between the sum of the first five terms and the limiting sum:

$T_n = ar^{n-1}$

$6.4 = a(0.8)^2$

$a = \frac{6.4}{(0.8)^2} = 10$

$S_5 = \frac{10(1 - 0.8^5)}{1 - 0.8}$

$= 33.616$

$S_\infty = \frac{10}{1 - 0.8} = 50$

$Diff. = 50 - 33.616$

$= 16.384$

Geometric Series

A geometric series is formed by multiplying the preceding term by a constant. The constant is called the **common ratio** and is assigned the letter *r*.

$$r = \frac{T_2}{T_1} = \frac{T_3}{T_2}$$

nth term of a Geometric Series

For a Geometric Series with first term *a*, common ratio *r*, T_n:

$$T_n = ar^{n-1}$$

Partial sum of a Geometric Series

The sum of the first *n* terms (*n*th partial sum):

$$S_n = \frac{a(r^n - 1)}{r - 1} for\ |r| > 1$$

$$S_n = \frac{a(1 - r^n)}{1 - r} for\ |r| < 1$$

Limiting Sum of a Geometric Series (Sum to infinity)

The sum of a GS with |r|<1, doesn't increase much after the first few terms, and is limited to a finite number (limiting sum):

$$S_\infty = \frac{a}{1 - r}$$