5.2 – Generating the terms of a first-order recurrence relation

A first-order recurrence relation defines a relationship between two successive terms of a sequence, for example, between:

u_n, **the previous term** u_{n+1}, **the next term.**

Another notation that can be used is:

u_{n-1}, **the previous term** u_n, **the next term.**

The first term can be represented by either u_0 **or** u_1.

Both notations may be used in SAC's or VCAA exams

WORKED EXAMPLE 1

The following equations each define a sequence. Which of them are first-order recurrence relations (defining a relationship between two consecutive terms)?

a. $u_n = u_{n-1} + 2 \qquad u_1 = 3 \qquad n = 1, 2, 3, \ldots$

b. $u_n = 4 + 2n \qquad n = 1, 2, 3, \ldots$

c. $f_{n+1} = 3f_n \qquad n = 1, 2, 3, \ldots$

THINK	WRITE
a The equation contains the consecutive terms u_n and u_{n-1} to describe a pattern with a known term.	**a** This is a first-order recurrence relation. It has the pattern $u_n = u_{n-1} + 2$ and a starting or first term of $u_1 = 3$.
b The equation contains only the u_n term. There is no u_{n+1} or u_{n-1} term.	**b** This is not a first-order recurrence relation because it does not describe the relationship between two consecutive terms.
c The equation contains the consecutive terms f_n and f_{n+1} to describe a pattern but has no known first or starting term.	**c** This is an incomplete first-order recurrence relation. It has no first or starting term, so a sequence cannot be commenced.

WORKED EXAMPLE 2

Write the first five terms of the sequence defined by the first-order recurrence relation:

$$u_n = 3u_{n-1} + 5 \qquad u_0 = 2.$$

THINK	WRITE
1 Since we know the u_0 or starting term, we can generate the next term, u_1, using the pattern: The next term is $3 \times$ the previous term $+ 5$.	$u_n = 3u_{n-1} + 5 \qquad u_0 = 2$ $u_1 = 3u_0 + 5$ $= 3 \times 2 + 5$ $= 11$
2 Now we can continue generating the next term, u_2, and so on.	$u_2 = 3u_1 + 5$ $= 3 \times 11 + 5$ $= 38$ $u_3 = 3u_2 + 5$ $= 3 \times 38 + 5$ $= 119$ $u_4 = 3u_3 + 5$ $= 3 \times 119 + 5$ $= 362$
3 Write your answer.	The sequence is $2, 11, 38, 119, 362$.

WORKED EXAMPLE 3

A sequence is defined by the first-order recurrence relation:

$$u_{n+1} = 2u_n - 3 \qquad n = 1, 2, 3, \ldots$$

If the fourth term of the sequence is -29, that is, $u_4 = -29$, then what is the second term?

THINK	WRITE
1 Transpose the equation to make the previous term, u_n, the subject.	$u_n = \dfrac{u_{n+1} + 3}{2}$
2 Use u_4 to find u_3 by substituting into the transposed equation.	$u_3 = \dfrac{u_4 + 3}{2} = \dfrac{-29 + 3}{2} = -13$
3 Use u_3 to find u_2.	$u_2 = \dfrac{u_3 + 3}{2} = \dfrac{-13 + 3}{2} = -5$
4 Write your answer.	The second term, u_2, is -5.

- ***5.3 – First-order linear recurrence relations***
  - **COMMON DIFFERENCE**

A sequence with a common difference of d may be defined by a first-order linear recurrence relation of the form:

$$u_{n+1} = u_n + d \qquad (\text{or } u_{n+1} - u_n = d)$$

where d is the common difference and for

$d > 0$ **it is an increasing sequence**

$d < 0$ **it is a decreasing sequence.**

WORKED EXAMPLE 4

Express each of the following sequences as first-order recurrence relations.

a. $7, 12, 17, 22, 27, \ldots$

b. $9, 3, -3, -9, -15, \ldots$

THINK	WRITE
a 1 Write the sequence.	**a** $7, 12, 17, 22, 27, \ldots$
2 Check for a common difference.	$d = u_4 - u_3 = 22 - 17 = 5$; $d = u_3 - u_2 = 17 - 12 = 5$; $d = u_2 - u_1 = 12 - 7 = 5$
3 There is a common difference of 5 and the first term is 7.	The first-order recurrence relation is given by: $u_{n+1} = u_n + d$; $u_{n+1} = u_n + 5 \quad u_1 = 7$
b 1 Write the sequence.	**b** $9, 3, -3, -9, -15, \ldots$
2 Check for a common difference.	$d = u_4 - u_3 = -9 - -3 = -6$; $d = u_3 - u_2 = -3 - 3 = -6$; $d = u_2 - u_1 = 3 - 9 = -6$
3 There is a common difference of -6 and the first term is 9.	The first-order recurrence relation is given by: $u_{n+1} = u_n - 6 \quad u_1 = 9$

WORKED EXAMPLE 5

Express the following sequence as a first-order recurrence relation.

$$u_n = -3n - 2 \qquad n = 1, 2, 3, 4, 5, \ldots$$

THINK	WRITE
1 Generate the sequence using the given rule.	$n = 1, 2, 3, 4, 5, \ldots$ $\quad u_n = -3n - 2$ $n = 1$ $\quad u_1 = -3 \times 1 - 2 = -3-2 = -5$ $n = 2$ $\quad u_2 = -3 \times 2 - 2 = -6 - 2 = -8$ $n = 3$ $\quad u_3 = -3 \times 3 - 2 = -9 - 2 = -11$ $n = 4$ $\quad u_4 = -3 \times 4 - 2 = -12 - 2 = -14$ The sequence is $-5, -8, -11, -14, \ldots$
2 There is a common difference of -3 and the first term is -5. Write the first-order recurrence relation.	The first order difference equation is: $u_{n+1} = u_n - 3 \quad u_1 = -5$

- **COMMON RATIO**

A sequence with a common ratio of R may be defined by a first-order linear recurrence relation of the form:

$$u_{n+1} = \mathrm{R}u_n$$

where R is the common ratio

$R > 1$ **is an increasing sequence**

$0 < R < 1$ **is a decreasing sequence**

$R < 0$ **is a sequence alternating between positive and negative values.**

WORKED EXAMPLE 6

Express each of the following sequences as first-order recurrence relations.

a. $1, 5, 25, 125, 625, \ldots$

b. $3, -6, 12, -24, 48 \ldots$

THINK	WRITE
a 1 There is a common ratio of 5 and the first term is 1.	**a** $R = \frac{5}{1} = \frac{25}{5} = \frac{125}{25} = \ldots = 5$ $u_1 = 1$
2 Write the first-order recurrence relation.	The first-order recurrence relation is given by: $u_{n+1} = 5u_n \quad u_1 = 1$
b 1 There is a common ratio of -2 and the first term is 3.	**b** $R = \frac{-6}{3} = \frac{12}{-6} = \frac{-24}{12} = \ldots = -2$ $u_1 = 3$
2 Write the first-order recurrence relation.	The first-order recurrence relation is given by: $u_{n+1} = -2u_n \quad u_1 = 3$

WORKED EXAMPLE 7

Express each of the following sequences as first-order recurrence relations.

a. $u_n = 2(7)^{n-1} \qquad n = 1, 2, 3, 4, \ldots$

b. $u_n = -3(2)^{n-1} \qquad n = 1, 2, 3, 4, \ldots$

THINK	WRITE
a **1** Generate the sequence using the given rule.	**a** $n = 1, 2, 3, 4, \ldots \quad u_n = 2(7)^{n-1}$ $n = 1 \quad u_1 = 2(7)^{1-1} = 2 \times 7^0 = 2 \times 1 = 2$ $n = 2 \quad u_1 = 2(7)^{2-1} = 2 \times 7^1 = 2 \times 7 = 14$ $n = 3 \quad u_1 = 2(7)^{3-1} = 2 \times 7^2 = 2 \times 49 = 98$ $n = 4 \quad u_1 = 2(7)^{4-1} = 2 \times 7^3 = 2 \times 343 = 686$
2 There is a common ratio of 7 and the first term is 2.	$R = 7, u_1 = 2$
3 Write the first-order recurrence relation.	The first-order recurrence relation is given by: $u_{n+1} = 7u_n \qquad u_1 = 2$
b **1** Generate the sequence using the given rule.	**b** $n = 1, 2, 3, 4, \ldots \quad u_n = -3(2)^{n-1}$ $n = 1 \quad u_1 = -3(2)^{1-1} = -3 \times 2^0 = -3 \times 1 = -3$ $n = 2 \quad u_1 = -3(2)^{2-1} = -3 \times 2^1 = -3 \times 2 = -6$ $n = 3 \quad u_1 = -3(2)^{3-1} = -3 \times 2^2 = -3 \times 4 = -12$ $n = 4 \quad u_1 = -3(2)^{4-1} = -3 \times 2^3 = -3 \times 8 = -24$
2 There is a common ratio of 2 and the first term is -3.	$R = 2, u_1 = -3$
3 Write the first-order recurrence relation.	The first-order recurrence relation is given by: $u_{n+1} = 2u_n \qquad u_1 = -3$

○ 5.4 – Graphs of first-order recurrence relations

○ ARITHMETIC PATTERNS (Common Difference)

-Distinguished by a constant increase or decrease.

WORKED EXAMPLE 8

On a graph, show the first five terms of the sequence described by the first-order recurrence relation:

$$u_{n+1} = u_n - 3 \qquad u_1 = -5.$$

THINK

1 Generate the values of each of the five terms of the sequence.

WRITE/DRAW

$u_{n+1} = u_n - 3 \qquad u_1 = -5$

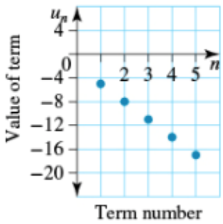
$u_2 = u_1 - 3 = -5 - 3 = -8$

$u_3 = u_2 - 3 = -8 - 3 = -11$

$u_4 = u_3 - 3 = -11 - 3 = -14$

$u_5 = u_4 - 3 = -14 - 3 = -17$

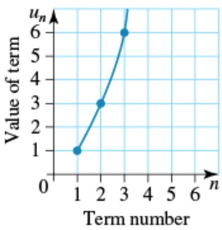
2 Graph these first five terms. The value of the term is plotted on the y-axis, and the term number is plotted on the x-axis.



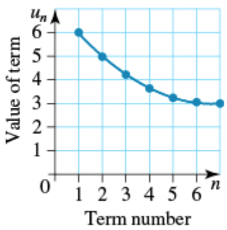
○ GEOMETRIC PATTERNS (Common Ratio)

First-order recurrence relations: $u_{n+1} = Ru_n$

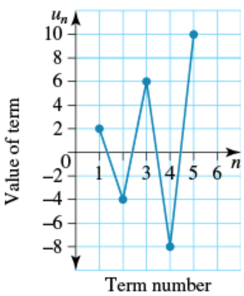
The sequences of a first-order recurrence relation $u_{n+1} = Ru_n$ are distinguished by a curved line or a saw form.



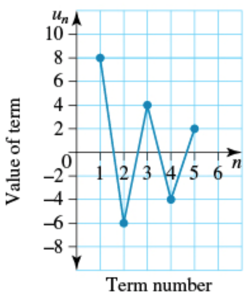
An increasing pattern or a positive common ratio greater than 1 ($R > 1$) gives an upward curved line.



A decreasing pattern or a positive fractional common ratio ($0 < R < 1$) gives a downward curved line.



An increasing saw pattern occurs when the common ratio is a negative value less than −1 ($R < -1$).



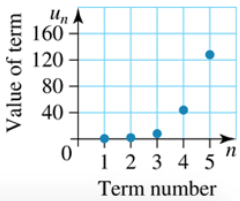
A decreasing saw pattern occurs when the common ratio is a negative fraction ($-1 < R < 0$).

-*Values of n in recurrence relations are integer values, so they should NOT be joined by lines.*

WORKED EXAMPLE 9

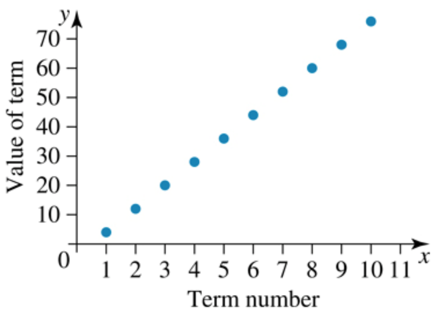
On a graph, show the first six terms of the sequence described by the first-order recurrence relation:

$$u_{n+1} = 4u_n \qquad u_1 = 0.5.$$

THINK	WRITE/DRAW
1 Generate the six terms of the sequence.	$u_{n+1} = 4u_n$ $u_1 = 0.5$ $u_2 = 4u_1 = 4 \times 0.5 = 2$ $u_3 = 4u_2 = 4 \times 2 = 8$ $u_4 = 4u_3 = 4 \times 8 = 32$ $u_5 = 4u_4 = 4 \times 32 = 128$ $u_6 = 4u_5 = 4 \times 128 = 512$
2 Graph these terms. *Note:* The sixth term is not included in this graph to more clearly illustrate the relationship between the terms.	

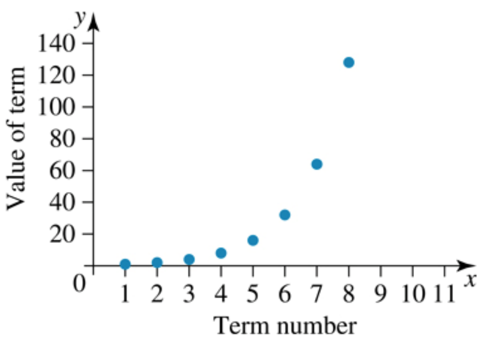
Interpretation of the graph of first-order recurrence relations

Straight or linear



A straight line or linear pattern is given by first-order recurrence relations of the form $u_{n+1} = u_n + d$

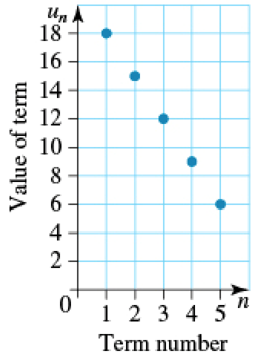
Non-linear (exponential)



A non-linear pattern is generated by first-order recurrence relations of the form $u_{n+1} = Ru_n$

WORKED EXAMPLE 10

The first five terms of a sequence are plotted on the graph. Write the first-order recurrence relation that defines this sequence.



THINK

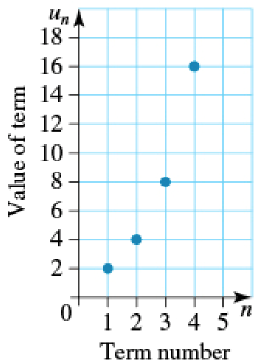
1. Read from the graph the first five terms of the sequence.
2. Notice that the graph is linear and there is a common difference of -3 between each term.
3. Write your answer including the value of one of the terms (usually the first), as well as the rule defining the first order difference equation.

WRITE

1. The sequence from the graph is: $18, 15, 12, 9, 6, \ldots$
2. $u_{n+1} = u_n + d$
   Common difference, $d = -3$
   $u_{n+1} = u_n - 3$ $\quad$ (or $u_{n+1} - u_n = -3$)
3. $u_{n+1} = u_n - 3 \quad u_1 = 18$

WORKED EXAMPLE 11

The first four terms of a sequence are plotted on the graph. Write the first-order recurrence relation that defines this sequence.



THINK

1. Read the terms of the sequence from the graph.
2. The graph is non-linear and there is a common ratio of 2, that is, for the next term, multiply the previous term by 2.
3. Define the first term.
4. Write your answer.

WRITE

1. The sequence is $2, 4, 8, 16, \ldots$
2. $u_{n+1} = R \times u_n$
   Common ratio, $R = 2$
   $u_{n+1} = 2u_n$
3. $u_1 = 2$
4. $u_{n+1} = 2u_n \quad u_1 = 2$

WORKED EXAMPLE 12

The first five terms of a sequence are plotted on the graph shown. Which of the following first-order recurrence relations could describe the sequence?

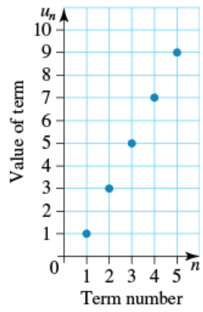
A. $u_{n+1} = u_n + 1$ with $u_1 = 1$

B. $u_{n+1} = u_n + 2$ with $u_1 = 1$

C. $u_{n+1} = 2u_n$ with $u_1 = 1$

D. $u_{n+1} = u_n + 1$ with $u_1 = 2$

E. $u_{n+1} = u_n + 2$ with $u_1 = 2$



THINK	WRITE
1 Eliminate the options systematically. Examine the first term given by the graph to decide if it is $u_1 = 1$ or $u_1 = 2$.	The coordinates of the first point on the graph are $(1, 1)$. The first term is $u_1 = 1$. Eliminate options D and E.
2 Observe any pattern between each successive point on the graph.	There is a common difference of 2 or $u_{n+1} = u_n + 2$.
3 Option B gives both the correct pattern and first term.	The answer is B.

○ 6.2 – Simple Interest

-Simple interest can be represented by the following first-order linear recurrence relation:

$$V_{n+1} = V_n + d, d = \frac{V_0 \times r}{100},$$

where V_n represents the value of the investment after n time periods, d is the amount of interest earned per period, V_0 is the initial (or starting) amount and r is the interest rate.

You can also calculate the total amount of a simple interest loan or investment by using:

$$\text{Total amount of loan or investment} = \text{initial amount or principal} + \text{interest}$$
$$V_n = V_0 + I$$

Simple interest is the percentage of the amount borrowed or invested multiplied by the number of time periods (usually years). The amount is added to the principal either as payment for the use of the money borrowed or as return on money invested.

$$I = \frac{V_0 r n}{100}$$

I = simple interest charged or earned (\$)

V_0 = principal (money invested or loaned) (\$)

r = rate of interest per period (% per period)

n = the number of periods (years, months, days) over which the agreement operates

WORKED EXAMPLE 1

$\$325$ is invested in a simple interest account for 5 years at 3% p.a. (per year).

a. Set up a recurrence relation to find the value of the investment after n years.

b. Use the recurrence relation from part a to find the value of the investment at the end of each of the first 5 years.

THINK

a 1 Write the formula to calculate the amount of interest earned per period (d).

2 List the values of V_0 and r.

3 Substitute into the formula and evaluate.

4 Use the values of d and V_0 to set up your recurrence relation.

b 1 Set up a table to find the value of the investment for up to $n = 5$.

2 Use the recurrence relation from part a to complete the table.

3 Write your answer.

WRITE

a $d = \dfrac{V_0 \times r}{100}$

$V_0 = 325, r = 3$

$d = \dfrac{325 \times 3}{100}$
$= 9.75$

$V_{n+1} = V_n + 9.75, V_0 = 325$

b

$n + 1$	V_n (\$)	V_{n+1} (\$)
1	325	$325 + 9.75 = 344.75$
2	334.75	$334.75 + 9.75 = 344.50$
3	334.50	$334.50 + 9.75 = 354.25$
4	354.25	$354.25 + 9.75 = 364$
5	364	$364 + 9.75 = 373.75$

The value of the investment at the end of each of the first 5 years is:
$\$334.75, \$344.50, \$354.25, \$364$ and $\$373.75$

WORKED EXAMPLE 2

Jan invested $\$210$ with a building society in a fixed deposit account that paid 8% p.a. simple interest for 18 months.

a. How much did she receive after the 18 months?

b. Represent the account balance for each of the 18 months graphically.

THINK

a 1 Write the simple interest formula.

2 List the values of V_0, r and n. Check that r and n are in the same time terms. Need to convert 18 months into years.

3 Substitute the values of the pronumerals into the formula and evaluate.

4 Write the answer.

5 Add the interest to the principal (total amount received).

6 Write your answer.

WRITE/DRAW

a $I = \dfrac{V_0 r n}{100}$

$V_0 = \$210$
$r = 8\%$ per year
$n = 18$ months
$= 1\frac{1}{2}$ years

$I = \dfrac{210 \times 8 \times 1.5}{100}$
$= 25.2$

The interest charged is $\$25.20$.

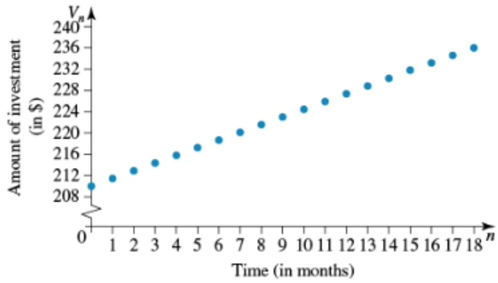
$V_n = V_0 + I$
$= 210 + 25.20$
$= 235.20$

Total amount received at the end of 18 months is $\$235.20$.

b 1 As we are dealing with simple interest, the value of the investment increases each month by the same amount. To find the monthly increase, divide the total interest earned by the number of months.

b
$$\text{Increase per month} = \frac{25.20}{18} = \$1.40$$

2 Draw a set of axes. Put time (in months) on the horizontal axis and the amount of investment (in \$) on the vertical axis. Plot the points: the initial value of investment is \$210 and it grows by \$1.40 each month. (The last point has coordinates (18, 235.20).)



○ Finding Vo, *r* and *n*

WORKED EXAMPLE 3

A bank offers 9% p.a. simple interest on an investment. At the end of 4 years the total interest earned was \$215. How much was invested?

THINK	WRITE
1 Write the simple interest formula.	$I = \frac{V_0 r n}{100}$
2 List the values of I, r and n. Check that r and n are in the same time terms.	$I = \$215$ $r = 9\%$ per year $n = 4$ years
3 Substitute into the formula.	$I = \frac{V_0 \times r \times n}{100}$ $215 = \frac{V_0 \times 9 \times 4}{100}$
4 Make V_0 the subject by multiplying both sides by 100 and dividing both sides by (9×4).	$V_0 = \frac{215 \times 100}{9 \times 4} = 597.22$
5 Write your answer in the correct units.	The amount invested was \$597.22.

Transposed simple interest formula

It may be easier to use the transposed formula when finding V_0, r or n.

To find the principal: $$V_0 = \frac{100 \times I}{r \times n}$$

To find the interest rate: $$r = \frac{100 \times I}{V_0 \times n}$$

To find the period of the loan or investment: $$n = \frac{100 \times I}{V_0 \times r}$$

WORKED EXAMPLE 4

When $\$720$ is invested for 36 months it earns $\$205.20$ simple interest. Find the yearly interest rate.

THINK

1 Write the simple interest formula, where rate is the subject.

2 List the values of V_0, I and n. n must be expressed in years so that r can be evaluated in $\%$ per year.

3 Substitute into the formula and evaluate.

4 Write your answer.

WRITE

$$r = \frac{100 \times I}{V_0 \times n}$$

$$V_0 = \$720$$
$$I = \$205.20$$
$$n = 36 \text{ months}$$
$$= 3 \text{ years}$$

$$r = \frac{100 \times 205.20}{720 \times 3}$$
$$= 9.5$$

The interest rate offered is 9.5% per annum.

WORKED EXAMPLE 5

An amount of $\$255$ was invested at 8.5% p.a. How long will it take, to the nearest year, to earn $\$86.70$ in interest?

THINK

1 Write the simple interest formula, where time is the subject.

2 Substitute the values of V_0, I and r. The rate, r is expressed per annum so time, n will be evaluated in the same time terms, namely years.

3 Substitute into the formula and evaluate.

4 Write your answer.

WRITE

$$n = \frac{100 \times I}{V_0 \times r}$$

$$V_0 = \$225$$
$$I = \$86.70$$
$$r = 8.5\% \text{ p.a.}$$

$$n = \frac{100 \times 86.70}{255 \times 8.5}$$
$$= 4$$

It will take 4 years.

6.3 – Compound Interest Tables

WORKED EXAMPLE 6

Laura invested $\$2500$ for 5 years at an interest rate of 8% p.a. with interest compounded annually. Complete the table by calculating the values A, B, C, D, E and F.

Time period $(n+1)$	$V_n(\$)$	Interest ($)	V_{n+1} ($)
1	2500	A% of 2500 = 200	2700
2	B	8% of C = 216	D
3	2916	8% of 2916 = 233.28	3149.28
4	3149.28	8% of 3149.28 = 251.94	E
5	F	8% of 3401.22 = 272.10	3673.32

THINK	WRITE
A: The percentage interest per annum earned on the investment.	A = 8
B: The principal at the start of the second year is the balance at the end of the first.	B = 2700
C: 8% interest is earned on the principal at the start of the second year.	C = B = 2700
D: The balance is the sum of the principal at the start of the year and the interest earned.	D = 2700 + 216 = 2916
E: The final balance is the sum of the principal at the start of the 4th year and the interest earned.	E = 3149.28 + 251.94 = 3401.22
F: The principle at the start of the fifth year is the balance at the end of the fourth year.	F = 3401.22

6.4 - Compound Interest Formula

$$V_{n+1} = V_n R$$

where V_{n+1} is the amount of the investment 1 time period after V_n, R is the growth or compounding factor $\left(= 1+\frac{r}{100}\right)$ and r is interest rate per period.

This pattern can be expanded further to write the value of the investment in terms of the initial investment. This is known as the compound interest formula.

$$V_n = V_0 R^n \text{ where } \begin{aligned} V_n &= \text{final or total amount(\$)} \\ V_0 &= \text{principal(\$)} \\ R &= \text{growth or compounding factor } \left(= 1 + \frac{r}{100}\right) \\ r &= \text{interest rate per period} \\ n &= \text{number of interest-bearing periods} \end{aligned}$$

Note that the compound interest formula gives the *total amount* in an account, not just the interest earned as in the simple interest formula.

To find the total interest compounded, I:

$$I = V_n - V_0 \quad \text{where} \quad \begin{aligned} V_n &= \text{final or total amount (\$)} \\ V_0 &= \text{principal (\$)} \end{aligned}$$

studyon

Unit 3	Compound interest
AOS R & FM	Concept summary
Topic 2	Practice questions
Concept 2	

-If compound interest is used, when plotted, the values of the investment at the end of each period form an exponential curve.

WORKED EXAMPLE 7

$\$5000$ is invested for 4 years at 6.5% p.a., interest compounded annually.

a. Generate the compound interest formula for this investment.

b. Find the amount in the balance after 4 years and the interest earned over this period.

THINK	WRITE
a 1 Write the compound interest formula.	**a** $V_n = V_0\left(1 + \frac{r}{100}\right)^n$
2 List the values of n, r and P.	$n = 4$ $r = 6.5$ $V_0 = 5000$
3 Substitute the values into the formula.	$V_n = 5000\left(1 + \frac{6.5}{100}\right)^n$
4 Simplify.	$V_n = 5000(1.065)^n$
b 1 Substitute $n = 4$ into the formula.	**b** $V_4 = 5000(1.065)^4$
2 Evaluate correct to 2 decimal places.	$= \$6432.33$
3 Subtract the principal from the balance.	$I = V_4 - V_0$ $= 6432.33 - 5000$ $= \$1432.33$
4 Write your answer.	The amount of interest earned is $\$1432.33$ and the balance is $\$6432.33$.

- **NON ANNUAL COMPOUNDING**

Annually/Yearly=1/year

Bi-annually/Bi-yearly=2/year

Quarterly=4/year

Monthly=12/year

Fortnightly=26/year

Weekly=52/year

$$\text{Number of interest periods, } n = \text{number of years} \times \text{number of interest periods per year}$$

$$\text{Interest rate per period, } r = \frac{\text{nominal interest rate per annum}}{\text{number of interest periods per year}}$$

Note: Nominal interest rate per annum is simply the annual interest rate advertised by a financial institution.

WORKED EXAMPLE 8

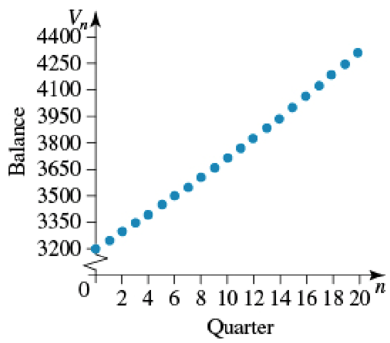
If $\$3200$ is invested for 5 years at 6% p.a., interest compounded quarterly:

a. find the number of interest-bearing periods, n

b. find the interest rate per period, r

c. find the balance of the account after 5 years

d. graphically represent the balance at the end of each quarter for 5 years. Describe the shape of the graph.

THINK	WRITE/DRAW
a Calculate n.	**a** $n = 5\text{ (years)} \times 4\text{(quarters)}$ $= 20$
b Convert % p.a. to % per quarter to match the time over which the interest is calculated. Divide $r\%$ p.a. by the number of compounding periods per year, namely 4. Write as a decimal.	**b** $r\% = \frac{6\%\text{ p.a.}}{4}$ $= 1.5\%$ per quarter; $r = 1.5$
c 1 Write the compound interest formula.	**c** $V_n = V_0\left(1 + \frac{r}{100}\right)^n$
2 List the values of V_0, r and n.	$V_0 = \$3200,\ r = 1.5,\ n = 20$
3 Substitute into the formula.	$V_{20} = \$3200\left(1 + \frac{1.5}{100}\right)^{20}$
4 Simplify.	$= 3200(1.015)^{20}$
5 Evaluate correct to 2 decimal places.	$= \$4309.94$
6 Write your answer.	Balance of account after 5 years is $\$4309.94$.
d 1 Using CAS, find the balance at the end of each quarter and plot these values on the set of axes. (The first point is $(0, 3200)$, which represents the principal.)	**d** 
2 Comment on the shape of the graph.	The graph is exponential as the interest is added at the end of each quarter and the following interest is calculated on the *new* balance.

WORKED EXAMPLE 9

Find the principal that will grow to $\$4000$ in 6 years, if interest is added quarterly at 6.5% p.a.

THINK	WRITE
1 Calculate n (there are 4 quarters in a year).	$n = 6 \times 4$ $= 24$
2 Calculate r.	$r = \frac{6.5}{4}$ $= 1.625$
3 List the value of V_{24}.	$V_{24} = 4000$
4 Write the compound interest formula, substitute and simplify.	$V_{24} = V_0\left(1 + \frac{r}{100}\right)^n$ $4000 = V_0\left(1 + \frac{1.625}{100}\right)^{24}$ $4000 = V_0(1.016\,25)^{24}$
5 Transpose to isolate V_0.	$V_0 = \frac{4000}{(1.016\,25)^{24}}$
6 Evaluate correct to 2 decimal places.	$= 2716.73$
7 Write a summary statement.	$\$2716.73$ would need to be invested.

○ *6.5 – Finding rate or time for Compound Interest*

-Using the compound interest formula we can calculate the interest that is needed to increase the value of our investment to the amount we desire.

-We must first find the interest rate/period, *r*, and convert this to the corresponding nominal rate/annum.

WORKED EXAMPLE 10

Find the interest rate per annum (correct to 2 decimal places) that would enable an investment of $\$4000$ over 2 years if interest is compounded quarterly.

THINK	WRITE
1 List the values of V_n, V_0 and n. For this example, n needs to represent quarters of a year and therefore r will be evaluated in $\%$ per quarter.	$V_n = \$4000$ $V_0 = \$3000$ $n = 2 \times 4$ $= 8$
2 Write the compound interest formula and substitute the known values.	$V_n = V_0R^n$ $4000 = 3000R^8$
3 Divide V_n by V_0	$\frac{4000}{3000} = R^8$
4 Obtain R to the power of 1, that is, raise both sides to the power of $\frac{1}{8}$.	$\left(\frac{4}{3}\right)^{\frac{1}{8}} = (R^8)^{\frac{1}{8}} = R$
5 Replace R with $1 + \frac{r}{100}$.	$\left(\frac{4}{3}\right)^{\frac{1}{8}} = 1 + \frac{r}{100}$

6 Isolate r and evaluate.	$\frac{r}{100} = \left(\frac{4}{3}\right)^{\frac{1}{8}} - 1$ $= 0.0366146$ $r = 3.661\ 46$ $r\% = 3.661\ 46\%$ per quarter
7 Multiply r by the number of interest periods per year to get the annual rate (correct to 2 decimal places).	Annual rate $= r\%$ per quarter $\times 4$ $= 3.661\ 46\%$ per quarter $\times 4$ $= 14.65\%$ per annum
8 Write your answer.	Interest rate of 14.65% p.a. is required, correct to 2 decimal places.

Note: Worked example 10 requires a number of operations to find the solution. This is one of the reasons why most financial institutions use finance software for efficient and error-free calculations. Your CAS has a finance function called **Finance Solver**. This can be used for compound interest calculations as shown in the worked examples in this section. Finance Solver will also be used extensively in the remaining sections of this topic.

-In this section, we will use the CAS Financial Solver to calculate the time period of an investment.

WORKED EXAMPLE 11

How long it will take $2000 to amount to $3500 at 8% p.a. with interest compounded annually?

THINK	WRITE
1 State the values of V_n, V_0 and r.	$V_n = \$3500$, $V_0 = \$2000$ and $r = 8\%$ p.a.
2 Use the Finance Solver on CAS to enter the following values: n (N:) = unknown r (I (%) :) = 8 V_0 (PV:) = −2000 V_n (FV:) = 3500 PpY: = 1 CpY: = 1	
3 Solve for n.	$n = 7.27$ years
4 Interest is compounded annually, so n represents years. Raise n to the next whole year.	As the interest is compounded annually, $n = 8$ years.
5 Write your answer.	It will take 8 years for $2000 to amount to $3500.

WORKED EXAMPLE 12

Calculate the number of interest-bearing periods, n, required and hence the time it will take $\$3600$ to amount to $\$5100$ at a rate of 7% p.a., with interest compounded quarterly.

THINK	WRITE
1 State the values of V_n, V_0 and r.	$V_n = \$5100$, $V_0 = \$3600$ and $r = 7\%$ p.a.
2 Enter the following values into the Finance Solver on your CAS: n (N:) = unknown r (I (%) :) = 7 V_0 (PV:) = −3600 V_n (FV:) = 5100 PpY: = 4 CpY: = 4	
3 Solve for n.	$n = 20.08$ quarters
4 Write your answer using more meaningful units.	As the interest is compounded annually, $n = 21$ quarters.
5 Answer the question fully.	It will take $5\frac{1}{4}$ years for $\$3600$ to amount to $\$5100$.

○ 6.6 – Flat Rate Depreciation/Prime Cost Depreciation

The estimated loss in value of assets is called **depreciation**. Each financial year a business will set aside money equal to the depreciation of an item in order to cover the cost of the eventual replacement of that item. The estimated value of an item at any point in time is called its **future value** (or book value).

When the value becomes zero, the item is said to be *written off*. At the end of an item's useful or effective life (as a contributor to a company's income) its future value is then called its **scrap value**.

Future value = cost price − total depreciation to that time

When book value = $\$0$, then the item is said to be written off.

Scrap value is the book value of an item at the end of its useful life.

There are 3 methods by which depreciation can be calculated. They are:

1. flat rate depreciation
2. reducing balance depreciation
3. unit cost depreciation.

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Interactivity

Depreciation: flat rate, reducing balance, unit cost

-If an item is depreciated by the **flat rate method,** then it's value decreases by a fixed amount each unit time interval, generally/year.

-This depreciation value may be expressed in dollars or as a % of the original cost price.

-The **flat rate method** is an example of **straight line (linear) decay**.

-The relationship can be represented by the recurrence relation:

$$V_{n+1} = V_n - d$$

where V_n is the value of the asset after n depreciating periods and d is the depreciation each time period.

We can also look at the future value of an asset after n periods of depreciation, which can be calculated by:

$$V_n = V_0 - nd$$

We can use this relationship to analyse flat rate depreciation or we can use a depreciation schedule (table) which can then be used to draw a graph of future value against time. The schedule displays the future value after each unit time interval, that is:

Time, n	Depreciation, d	Future value, V_n

WORKED EXAMPLE 13

Fast Word Printing Company bought a new printing press for $\$15\ 000$ and chose to depreciate it by the flat rate method. The depreciation was 15% of the prime cost price each year and its useful life was 5 years.

a. Find the annual depreciation.

b. Set up a recurrence relation to represent the depreciation.

c. Draw a depreciation schedule for the useful life of the press and use it to draw a graph of book value against time.

d. Generate the relationship between the book value and time and use it to find the scrap value.

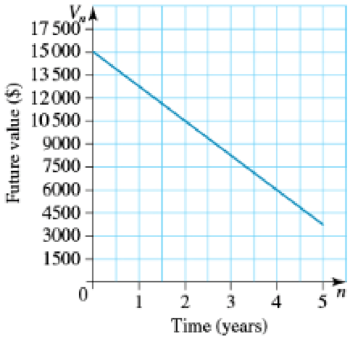
THINK			WRITE/DRAW
a	1	State the cost price.	a $V_0 = 15\ 000$
	2	Find the depreciation rate as 15% of the prime cost price.	$d = V_0 \times \frac{r}{100}$ $= 15\ 000 \times \frac{15}{100}$ $= 2250$
	3	Write your answer.	Annual depreciation is $\$2250$.
b		Write the recurrence relation for flat rate depreciation and substitute in the value for d.	b $V_{n+1} = V_n - d$ $V_{n+1} = V_n - 2250$

c **1** Draw a depreciation schedule for 0–5 years, using depreciation of 2250 each year and a starting value of $15 000.

c

Time, n (years)	Depreciation, d ($)	Future value, V_n ($)
0		15 000
1	2250	12 750
2	2250	10 500
3	2250	8 250
4	2250	6 000
5	2250	3 750

2 Draw a graph of the tabled values for future value against time.



d **1** Set up the equation: $V_n = V_0 = nd$. State d and V_0

d

$$d = 2250$$
$$V_0 = 15\ 000$$
$$V_n = 15\ 000 - 2250n$$

2 The press is scrapped after 5 years so substitute $n = 5$ into the equation.

$$V_5 = 15\ 000 - 2250(5)$$
$$= 15\ 000 - 11\ 250$$
$$= 3750$$

3 Write your answer.

The scrap value is $3750.

WORKED EXAMPLE 14

Jarrod bought his car 5 years ago for $15 000. Its current market value is $7500. Assuming straight line depreciation, find:

a. the car's annual depreciation rate

b. the relationship between the future value and time, and use it to find when the car will have a value of $3000.

THINK / **WRITE**

a **1** Find the total depreciation over the 5 years and thus the rate of depreciation.

a

$$\text{Total depreciation} = \text{cost price} - \text{current value}$$
$$= \$15\ 000 - \$7500$$
$$= \$7500$$
$$\text{Rate of depreciation} = \frac{\text{total depreciation}}{\text{number of years}}$$
$$= \frac{\$7500}{5 \text{ years}}$$
$$= \$1500 \text{ per year}$$

2 Write your answer.

The annual depreciation rate is $1500.

b **1** Set up the future value equation.

b $V_n = V_0 - nd$

$V_n = 15\,000 - 1500n$

2 Use the equation and substitute $V_n = \$3000$ and transpose the equation to find n.

When $V_n = 3000$,

$$3000 = 15\,000 - 1500n$$
$$-1500n = 3000 - 15\,000$$
$$-1500n = -12\,000$$
$$n = \frac{-12\,000}{-1500}$$
$$= 8$$

3 Write your answer.

The depreciation equation for the car is $V_n = 15\,000 - 1500n$. The future value will reach $\$3000$ when the car is 8 years old.

○ 6.7 – Reducing Balance Depreciation

-If an item depreciates by the **reducing balance depreciation** method then its value decreases by a fixed rate/unit time interval/year.

-This rate is a % of the previous value of the item.

Reducing balance depreciation is also known as *diminishing value depreciation*.

Reducing balance depreciation can be expressed by the recurrence relation:

$$V_{n+1} = RV_n$$

where V_n is the value of the asset after n depreciating periods and $R = 1 - \frac{r}{100}$, where r is the depreciation rate.

WORKED EXAMPLE 15

Suppose the new $\$15\,000$ printing press considered in Worked example 13 was depreciated by the reducing balance method at a rate of 20% p.a. of the previous value.

a. Generate a depreciation schedule using a recurrence relation for the first 5 years of work for the press.

b. What is the future value after 5 years?

c. Draw a graph of future value against time.

THINK / **WRITE/DRAW**

a **1** Calculate the value of R.

a $R = 1 - \frac{r}{100}$

$= 1 - \frac{20}{100}$

$= 0.8$

2 Write the recurrence relation for reducing balance depreciation and substitute in the known information.

$V_{n+1} = RV_n$

$V_{n+1} = 0.8V_n,\ V_0 = 15\,000$

3 Use the recurrence relation to calculate the future value for the first 5 years (up to $n = 5$).

$$V_1 = 0.8V_0 = 0.8 \times 15\,000 = 12\,000$$
$$V_2 = 0.8V_1 = 0.8 \times 12\,000 = 9600$$
$$V_3 = 0.8V_2 = 0.8 \times 9600 = 7680$$
$$V_4 = 0.8V_3 = 0.8 \times 7680 = 6144$$
$$V_5 = 0.8V_4 = 0.8 \times 6144 = 4915.2$$

4 Draw the depreciation schedule.

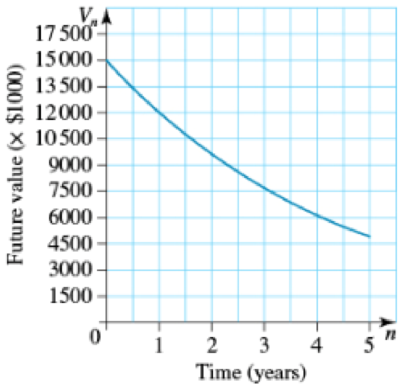
Time, n (years)	Future value, V_n ($)
0	15 000
1	12 000
2	9600
3	7680
4	6144
5	4915.20

b State the future value after 5 years from the depreciation schedule.

b The future value of the press after 5 years will be $4915.20.

c Draw a graph of the future value against time.

c



-Let us compare the **flat rate method** and the **reducing balance method**.

WORKED EXAMPLE 16

A transport business has bought a new bus for $\$60\ 000$. The business has the choice of depreciating the bus by a flat rate of 20% of the cost price each year or by 30% of the previous value each year.

a. Generate depreciation schedules using both methods for a life of 5 years.

b. Draw graphs of the future value against time for both methods on the same set of axes.

c. After how many years does the reducing balance future value become greater than the flat rate future value?

THINK

a 1 Calculate the flat rate depreciation per year.

2 Generate a flat rate depreciation schedule for 0–5 years.

3 Generate a reducing balance depreciation schedule. Annual depreciation is 30% of the previous value, so to calculate the future value multiply the previous value by 0.7. Continue to calculate the future value for a period of 5 years.

b Draw graphs using values for V_n and n from the schedules. In this instance the blue line is the flat rate value and the pink curve is the reducing balance value.

c Look at the graph to see when the reducing balance curve lies above the flat rate line. State the first whole year after this point of intersection.

WRITE/DRAW

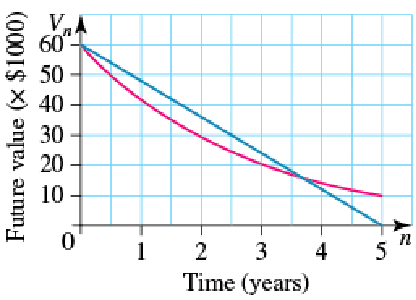
a $d = 20\%$ of $\$60\ 000$

$= \$12\ 000$ per year

Time, n (years)	Depreciation, d (\$)	Future value, V_n (\$)
0	—	60 000
1	12 000	48 000
2	12 000	36 000
3	12 000	24 000
4	12 000	12 000
5	12 000	0

Time, n (years)	Future value, V_n (\$)
0	60 000
1	$60\ 000 \times 0.7 = 42\ 000$
2	$42\ 000 \times 0.7 = 29\ 400$
3	$29\ 400 \times 0.7 = 20\ 580$
4	$20\ 580 \times 0.7 = 14\ 406$
5	$14\ 406 \times 0.7 = 10\ 084.20$

b



c The future value for the reducing balance method is greater than that of the flat rate method after 4 years.

We can write a general formula for reducing balance depreciation which is similar to the compound interest formula, except that the rate is subtracted rather than added to 1.

The reducing balance depreciation formula is:

$V_n = V_0 R^n$

V_n = book value after time, n

R = rate of depreciation $\left(= 1 - \frac{r}{100}\right)$

V_0 = cost price

n = time since purchase

WORKED EXAMPLE 17

The printing press from Worked example 13 was depreciated by the reducing balance method at 20% p.a. What will be the future value and total depreciation of the press after 5 years if it cost $\$15\ 000$ new?

THINK	WRITE
1 State V_0, r and n.	$V_0 = 15\ 000, r = 20, n = 5$
2 Calculate the value of R.	$R = 1 - \frac{20}{100} = 0.8$
3 Substitute into the depreciation formula and simplify.	$V_n = V_0 R^n$ $V_5 = 15\ 000(0.8)^5$
4 Evaluate.	$= 4915.2$
5 Total depreciation is: cost price – future value.	Total depreciation $= V_0 - V_5$ $= 15\ 000 - 4915.2$ $= 10\ 084.8$
6 Write a summary statement.	The future value of the press after 5 years will be $\$4915.20$ and its total depreciation will be $\$10\ 084.80$.

Effective life

The situation may arise where the scrap value is known and we want to know how long it will be before an item reaches this value; that is, its useful or **effective life**.

So, in the reducing balance formula $V_n = V_0 R^n$, n is needed.

WORKED EXAMPLE 18

A photocopier purchased for $\$8000$ depreciates by 25% p.a. by the reducing balance method. If the photocopier has a scrap value of $\$1200$, how long will it be before this value is reached?

THINK	WRITE
1 State the values of V_n, V_0 and r.	$V_n = \$1200$, $V_0 = \$8000$ and $r = 25\%$
2 Calculate the value of R.	$R = 1 - \frac{25}{100}$ $= 0.75$
3 Substitute the values of the pronumerals into the formula and simplify.	$V_n = V_0 R^n$ $1200 = 8000 \times (0.75)^n$ $0.15 = (0.75)^n$
4 Use CAS to find the value of n.	$n = 6.59$ years
5 Interest is compounded annually, so n represents years. Raise n to the next whole year.	As the depreciation is calculated once a year, $n = 7$ years.
6 Write your answer.	It will take 7 years for the photocopier to reach its scrap value.

○ 6.8 – Unit Cost Depreciation

The future value over time using unit cost depreciation can be expressed by the recurrence relation:

$$V_{n+1} = V_n - d$$

where V_n is the value of the asset after n outputs and d is the depreciation per output.

WORKED EXAMPLE 19

A motorbike purchased for $\$12\,000$ depreciates at a rate of $\$14$ per 100 km driven.

a. Set up a recurrence relation to represent the depreciation.

b. Use the recurrence relation to generate a depreciation schedule for the future value of the bike after it has been driven for 100 km, 200 km, 300 km, 400 km and 500 km.

THINK	WRITE
a 1 Write the recurrence relation for unit cost depreciation as well as the known information.	**a** $V_{n+1} = V_n - d$ $V_0 = 12\,000$, $d = 14$
2 Substitute the values into the recurrence relation.	$V_{n+1} = V_n - 14$, $V_0 = 12\,000$

b 1 Use the recurrence relation to calculate the future value for the first 5 outputs (up to 500 km).

b

$$\begin{aligned} V_1 &= V_0 - 14 \\ &= 12\,000 - 14 \\ &= 11\,986 \\ V_2 &= V_1 - 14 \\ &= 11\,986 - 14 \\ &= 11\,972 \\ V_3 &= V_2 - 14 \\ &= 11\,972 - 14 \\ &= 11\,958 \\ V_4 &= V_3 - 14 \\ &= 11\,958 - 14 \\ &= 11\,944 \\ V_5 &= V_4 - 14 \\ &= 11\,944 - 14 \\ &= 11\,930 \end{aligned}$$

2 Draw the depreciation schedule.

Distance driven (km)	Outputs (n)	Future value, V_n (\$)
100	1	11 986
200	2	11 972
300	3	11 958
400	4	11 944
500	5	11 930

WORKED EXAMPLE 20

A taxi is bought for \$31 000 and it depreciates by 28.4 cents per kilometre driven. In one year the car is driven 15 614 km. Find:

a. the annual depreciation for this particular year

b. its useful life if its scrap value is \$12 000.

THINK / **WRITE**

a 1 Depreciation amount = distance travelled × rate

a

$$\begin{aligned} \text{depreciation} &= 15\,614 \times \$0.284 \\ &= \$4434.38 \end{aligned}$$

2 Write a summary statement.

Annual depreciation for the year is \$4434.38.

b 1 Total depreciation = cost price − scrap value

Distance travelled $= \dfrac{\text{total depreciation}}{\text{rate of depreciation}}$

where rate of depreciation

= 28.4 cents/km
= \$0.284 per km

b

$$\begin{aligned} \text{Total depreciation} &= 31\,000 - 12\,000 \\ &= \$19\,000 \end{aligned}$$

$$\begin{aligned} \text{Distance travelled} &= \frac{19\,000}{0.284} \\ &= 66\,901 \text{ km} \end{aligned}$$

2 State your answer.

The taxi has a useful life of 66 901 km.

WORKED EXAMPLE 21

A photocopier purchased for $\$10\ 800$ depreciates at a rate of 20 cents for every 100 copies made. In its first year of use $500\ 000$ copies were made and in its second year, $550\ 000$. Find:

a. the depreciation each year

b. the future value at the end of the second year.

THINK

a To find the depreciation, identify the rate and number of copies made.

Express the rate of 20 cents per 100 copies in a simpler form of dollars per 100 copies, that is, $\$0.20$ per 100 copies or $\frac{0.20}{100 \text{ copies}}$.

b $\text{Future value} = \text{cost price} - \text{total depreciation}$

WRITE

a $\text{Depreciation} = \text{copies made} \times \text{rate}$

$$\text{depreciation}_{1\text{ st year}} = 500\ 000 \times \frac{0.20}{100 \text{ copies}} = \$1000$$

Depreciation in the first year is $\$1000$.

$$\text{depreciation}_{2\text{nd year}} = 550\ 000 \times \frac{0.20}{100 \text{ copies}} = \$1100$$

Depreciation in the second year is $\$1100$.

b Total depreciation after 2 years

$$= 1000 + 1100$$
$$= \$2100$$

$$\text{Book value} = 10\ 800 - 2100 = \$8700$$

The future value after n outputs using unit cost depreciation can be expressed as:

$$V_n = V_0 - nd$$

where V_n is the value of the asset after n outputs and d is the depreciation per output.

WORKED EXAMPLE 22

The initial cost of a vehicle was $\$27\ 850$ and its scrap value is $\$5050$. If the vehicle needs to be replaced after travelling $80\ 000$ km (useful life):

a. find the depreciation rate (depreciation (\$) per km)

b. find the amount of depreciation in a year when $16\ 497$ km were travelled

c. set up an equation to determine the value of the car after travelling n km

d. find the future value after it has been used for a total of $60\ 000$ km

e. set up a schedule table listing future value for every $20\ 000$ km.

THINK		WRITE
a	**1** To find the depreciation rate, first find the total depreciation. Total amount of depreciation = cost price − scrap value	**a** Total amount of depreciation $= 27\ 850 - 5050$ $= \$22\ 800$
	2 Find the rate of depreciation. It is common to express rates in cents per use if less than a dollar.	$\text{Depreciation rate} = \frac{\text{total depreciation}}{\text{total distance travelled}} = \frac{22\ 800}{80\ 000} = \$0.285 \text{ per km} = 28.5 \text{ cents per km}$
b	Find the amount of depreciation using the rate calculated. Amount of depreciation is always expressed in dollars.	**b** Amount of depreciation $= \text{amount of use} \times \text{rate of depreciation} = 16\ 497 \times 28.5 = 470\ 165 \text{ cents} = \$4701.65$
c	**1** Write the equation for unit cost depreciation after n outputs as well as the known information.	**c** $V_n = V_0 - \text{nd}$; $V_0 = 27\ 850,\ d = 0.285$
	2 Substitute the values into the equation.	$V_n = 27\ 850 - 0.285d$
d	**1** Use the equation from part c to find the future value when $n = 60\ 000$.	**d** $V_{60\ 000} = 27\ 850 - 0.285 \times 60\ 000 = 27\ 850 - 17100 = 10\ 750$
	2 Write your answer.	The future value after the car has been used for $60\ 000$ km is $\$10\ 750$.

e Calculate the future value for every 20 000 km of use and summarise in a table.

e

Use, n (km)	Future value, V_n ($)
0	$27 850
20 000	$V_1 = 27\,850 - (20\,000 \times 0.285)$ $= \$22\,150$
40 000	$V_2 = 27\,850 - (20\,000 \times 0.285 \times 2)$ $= \$16\,450$
60 000	$V_3 = 27\,850 - (20\,000 \times 0.285 \times 3)$ $= \$10\,750$
80 000	$V_4 = 27\,850 - (20\,000 \times 0.285 \times 4)$ $= \$5\,050$

○ ***7.2 – Reducing Balance Loans I***

Annuities

An annuity is an investment or a loan that has regular and constant payments over a period of time. Let's first look at this as a recurrence relation that calculates the value of an annuity after each time period.

$$V_{n+1} = V_n R - d$$

where:

V_{n+1} = **Amount left after** $n + 1$ **payments**

V_n = **Amount at time** n

$R = \left(1 + \frac{r}{100}\right)$, **where** r **is the interest rate per period**

d = **Payment amount**

-Less of the amount borrowed is paid off in the early stages of the loan compared to that of the end.

-Principal, Vo = amount borrowed ($)

-Balance, Vn = amount still owing ($)

-Term = life of the loan (years)

-To discharge a loan = to pay off a loan.

-Regular payments are called **annuities**.

WORKED EXAMPLE 1

A loan of $\$100\ 000$ is taken out over 15 years at a rate of 7.5% p.a. (interest debited monthly) and is to be paid back monthly with $\$927$ instalments. Complete the table below for the first five payments.

$n+1$	V_n	d	V_{n+1}
1	\$100 000		
2			
3			
4			
5			

THINK / **WRITE**

1 State the initial values for V_0, r and d.

$$V_0 = \$100\ 000$$
$$r = \frac{7.5}{12} = 0.625$$
$$d = \$927$$

2 Evaluate V_1.

$$V_{n+1} = V_n\left(1 + \frac{r}{100}\right) - d$$
$$V_1 = 100\ 000\left(1 + \frac{0.625}{100}\right) - 927 = \$99\ 698$$

3 Evaluate V_2.

$$V_{n+1} = V_n\left(1 + \frac{r}{100}\right) - d$$
$$V_2 = 99\ 698\left(1 + \frac{0.625}{100}\right) - 927 = \$99\ 394.11$$

4 Evaluate V_3.

$$V_{n+1} = V_n\left(1 + \frac{r}{100}\right) - d$$
$$V_3 = 99\ 394.11\left(1 + \frac{0.625}{100}\right) - 927 = \$99\ 088.32$$

5 Evaluate V_4.

$$V_{n+1} = V_n\left(1 + \frac{r}{100}\right) - d$$
$$V_4 = 99\ 088.32\left(1 + \frac{0.625}{100}\right) - 927 = \$98\ 780.62$$

6 Evaluate V_5.

$$V_{n+1} = V_n\left(1 + \frac{r}{100}\right) - d$$
$$V_5 = 98\ 780.62\left(1 + \frac{0.625}{100}\right) - 927 = \$98\ 471.00$$

7 Complete the table.

$n+1$	V_n	d	V_{n+1}
1	$V_0 = \$100\ 000$	\$927	$V_1 = \$99\ 698$
2	$V_1 = \$99\ 698$	\$927	$V_2 = \$99\ 394.11$
3	$V_2 = \$99\ 394.11$	\$927	$V_3 = \$99\ 088.32$
4	$V_3 = \$99\ 088.32$	\$927	$V_4 = \$98\ 780.62$
5	$V_4 = \$98\ 780.62$	\$927	$V_5 = \$98\ 471.00$

The annuities formula

The annuities formula can be used to find the amount still owing at any point in time during the term of a reducing balance loan. When a consumer borrows money from a financial institution, that person contracts to make regular payments or annuities in order to repay the sum borrowed over time.

The amount owing in a loan account for n repayments is given by the annuities formula:

$$V_n = V_0R^n - \frac{d(R^n - 1)}{R - 1}$$

where:

$V_0 =$ **the amount borrowed (principal)**

$R =$ **the compounding or growth factor for the amount borrowed**

$= 1 + \frac{r}{100}$ **(** $r =$ **the interest rate per repayment period)**

$d =$ **the amount of the regular payments made per period**

$n =$ **the number of payments**

$V_n =$ **the amount owing after** n **payments**

WORKED EXAMPLE 2

A loan of $\$50\ 000$ is taken out over 20 years at a rate of 6% p.a. (interest debited monthly) and is to be repaid with monthly instalments of $\$358.22$. Find the amount still owing after 10 years.

THINK	WRITE
1 State the loan amount, V_0, and the regular repayment, d.	$V_0 = 50\ 000$ $d = 358.22$
2 Find the number of payments, n, the interest rate per month, r, and the growth factor, R.	$n = 10 \times 12 = 120$ $r = \frac{6}{12} = 0.5$ $R = 1 + \frac{r}{100} = 1.005$
3 Substitute into the annuities formula.	$V_{120} = V_0R^n - \frac{d(R^n - 1)}{R - 1}$ $= 50\ 000(1.005)^{120} - \frac{358.22(1.005^{120} - 1)}{1.005 - 1}$
4 Evaluate V_{120}.	$V_{120} = \$32\ 264.98$
5 Write a statement.	The amount still owing after 10 years will be $\$32\ 264.98$

WORKED EXAMPLE 3

Rob wants to borrow $\$2800$ for a new sound system at 7.5% p.a., interest adjusted monthly.

a. What would be Rob's monthly repayment if the loan is fully repaid in $1\frac{1}{2}$ years?

b. What would be the total interest charged?

THINK	WRITE
a 1 State the value of V_0, n, r and R.	**a** $V_0 = 2800$ $n = 18$ $r = \frac{7.5}{12} = 0.625$ $R = 1 + \frac{0.625}{100} = 1.006\,25$
2 Substitute into the annuities formula to find the regular monthly repayment, d.	$d = \frac{V_0 R^n (R-1)}{R^n - 1} = \frac{2800(1.006\,25)^{18}(1.006\,25-1)}{1.006\,25^{18} - 1}$
3 Evaluate d.	$d = \$164.95$
4 Write a statement.	The monthly regular payment is $\$164.95$ over 18 months.
b 1 Total interest = total repayments − amount borrowed	**b** Total interest $= 164.95 \times 18 - 2800 = 2969.10 - 2800 = \$169.10$
2 Write a statement.	The total interest on the $\$2800$ loan over 18 months is $\$169.10$.

WORKED EXAMPLE 4

Josh borrows $\$12\,000$ for some home office equipment. He agrees to repay the loan over 4 years with monthly instalments at 7.8% p.a. (adjusted monthly). Find:

a. the instalment value

b. the principal repaid and interest paid during the:

i. 10th repayment

ii. 40th repayment.

THINK			WRITE
a	1	State the value of V_0, n, r and R.	a $V_0 = 12\,000$ $n = 4 \times 12 = 48$ $r = \frac{7.8}{12} = 0.65$ $R = 1 + \frac{0.65}{100} = 1.0065$
	2	Substitute into the annuities formula to find the monthly repayment, d.	$d = \frac{V_0 R^n (R-1)}{R^n - 1}$ $= \frac{12\,000(1.0065)^{48}(1.0065-1)}{1.0065^{48}-1}$
	3	Evaluate d.	$d = \$291.83$
	4	Write a statement.	The monthly repayment over a 4-year period is $\$291.83$.
b i	1	Find the amount owing after 9 months. (a) State V_0, n, R. (b) Substitute into the annuities formula.	b i $V_0 = 12\,000, n = 9, R = 1.0065$ $V_n = V_0 R^n - \frac{d(R^n - 1)}{R-1}$ $V_9 = 12\,000(1.0065)^9 - \frac{291.83(1.0065^9 - 1)}{1.0065 - 1}$
	2	Evaluate V_9.	$V_9 = \$10\,024.73$
	3	Find the amount owing after 10 months. Substitute (change $n = 9$ to $n = 10$) and evaluate.	$V_{10} = 12\,000(1.0065)^{10} - \frac{291.83(1.0065^{10} - 1)}{1.0065 - 1}$ $= \$9798.06$
	4	$\text{Principal repaid} = V_9 - V_{10}$	$\text{Principal repaid} = 10\,024.73 - 9798.06 = \$226.67$
	5	Interest paid = repayment − principal repaid	$\text{Total interest} = \$291.83 - 226.67 = \$65.16$
	6	Write a statement.	In the 10th repayment, $\$226.67$ principal is repaid and $\$65.16$ interest is paid.

ii **1** Repeat steps 1–6 for V_{39} and V_{40}.

ii

$$V_{39} = 12\,000(1.0065)^{39} - \frac{291.83(1.0065^{39} - 1)}{1.0065 - 1}$$

$$= \$2543.10$$

$$V_{40} = 12\,000(1.0065)^{40} - \frac{291.83(1.0065^{40} - 1)}{1.0065 - 1}$$

$$V_{40} = \$2267.80$$

$$\text{Principal repaid} = V_{39} - V_{40}$$

$$= 2543.10 - 2267.80$$

$$= \$275.30$$

$$\text{Interest} = 291.83 - 275.30$$

$$= \$16.53$$

2 Write a statement.

In the 40th repayment, $\$275.30$ principal is repaid
and $\$16.53$ interest is paid.

○ ***7.3 Reducing Balance Loans II***

Number of repayments

The situation often arises in reducing balance loans when a potential borrower knows how much needs to be borrowed as well as the amount that can be repaid each month. The person then wants to know how long the loan needs to be to accommodate these conditions, that is, to determine the number of repayments, n, required. As with compound interest, n is calculated using Finance Solver on your CAS.

WORKED EXAMPLE 5

A reducing balance loan of $\$60\ 000$ is to be repaid with monthly instalments of $\$483.36$ at an interest rate of 7.5% p.a. (debited monthly). Find:

a. the number of monthly repayments (and, hence, the term of the loan in more meaningful units) needed to repay the loan in full

b. the total interest charged.

THINK

a **1** Using Finance Solver on your CAS enter the appropriate values:

$n(\text{N:}) = \text{unknown}$

$r(\text{I}(\%)\text{:}) = 7.5$

$V_0(\text{PV:}) = 60\ 000$

$d(\text{Pmt:}) = -483.36$

$V_n(\text{FV:}) = 0$

$\text{PpY:} = 12 \text{ (monthly)}$

$\text{CpY:} = 12 \text{ (monthly)}$

WRITE

a $n = 239.995\ 307\ 038\ 33$

2 Solve for n. $n = 240$ months

3 Interpret the results.

$$\text{Time} = \frac{240}{12}\text{ years}$$
$$= 20 \text{ years}$$

4 Write a statement. Term of loan needed is 20 years.

b 1 Total interest = Total repayments − Principal repaid

b $$\text{Interest} = 483.36 \times 240 - 60\,000$$
$$= \$56\,006.40$$

2 Write a statement. Total interest charged on the loan is \$56 006.40.

Sometimes we may want to find the time for only part of the loan term. The procedure that is followed is the same as in Worked example 5; however, V_n is zero only if we are calculating the time to repay the loan in full. Otherwise we should consider the amount still owing at that time.

WORKED EXAMPLE 6

Some time ago, Petra borrowed \$14 000 to buy a car. Interest on this reducing balance loan has been charged at 9.2% p.a. (adjusted monthly) and she has been paying \$446.50 each month to service the loan. Currently she still owes \$9753.92. How long ago did Petra borrow the money?

THINK	WRITE
1 Identify V_n, V_0, d and r and enter the following values into Finance Solver on your CAS: $n(\text{N:}) = \text{unknown}$ $r(\text{I(\%):}) = 9.2$ $V_0(\text{PV:}) = 14\,000$ $d(\text{Pmt:}) = -446.50$ $V_n(\text{FV:}) = -9753.92$ $\text{PpY:} = 12$ (monthly payment) $\text{CpY:} = 12$ (monthly compounds)	$n = 11.999\,995\,037\,662$
2 Solve for n.	$n = 12$ months
3 Interpret the results.	Time = 1 years
4 Write a statement.	Petra has had the loan for the past 12 months.

In the situations investigated so far, we have considered calculating only the time from the start of the loan to a later date (including repayment in full). In fact, it doesn't matter what period of the loan is considered; we can still use Finance Solver as we have already done. In using CAS to do this, we can use V_n and V_0 such that they have the following meanings:

V_n = amount owing at the end of the time period

V_0 = amount owing at the start of the time period.

WORKED EXAMPLE 7

A loan of $\$11\ 000$ is being repaid by monthly instalments of $\$362.74$ with interest being charged at 11.5% p.a. (debited monthly). Currently, the amount owing is $\$7744.05$. How much longer will it take to:

a. reduce the amount outstanding to $\$2105.11$

b. repay the loan in full?

THINK	WRITE
a 1 Let V_0 be the amount still owing at the start of the time period. State V_n, V_0, d and r.	**a** $V_n = 2105.11$, $V_0 = 7744.05$, $Q = 362.74$, $r = 11.5\%$ p. a.
2 Using the Finance Solver on your CAS, enter the following values: n (N:) = unknown; r (I(%):) = 11.5; V_0 (PV:) = 7744.05; d (Pmt:) = −362.74; V_n (FV:) = −2105.11; PpY: = 12; CpY: = 12	$n = 17.999\ 988\ 603\ 29$
3 Solve for n.	$n = 18$ months
4 Interpret the results and write a statement.	It will take another $1\frac{1}{2}$ years to reduce the amount owing to $\$2105.11$.
b 1 Repeat part a, entering the appropriate values into Finance Solver. Enter FV: = 0 to represent the loan is fully repaid.	**b** $n = 23.999\ 534\ 856\ 457$ $= 24$ months
2 Write a statement.	It will take another 2 years to repay the loan in full.

○ EFFECTS OF CHANGING THE REPAYMENT

-In this section, we will look at the effect that changing the repayment value has on the term of the loan and the total interest paid.

WORKED EXAMPLE 8

A reducing balance loan of $\$16\ 000$ has a term of 5 years. It is to be repaid by monthly instalments at a rate of 8.4% p.a. (debited monthly).

a. Find the repayment value.

b. What will be the term of the loan if the repayment is increased to $\$393.62$?

c. Calculate the total interest paid for repayments of $\$393.62$.

d. By how much does the interest figure in c differ from that paid for the original offer?

THINK	WRITE
a **1** (a) Write the equation for d. (b) Give the values of V_0, n, r and R.	**a** $d = \dfrac{V_0 R^n (R-1)}{R^n - 1}$ $V_0 = 16\ 000$, $n = 5 \times 12 = 60$ $r = \dfrac{8.4}{12} = 0.7$ $R = 1.007$
2 Substitute into the annuities formula to evaluate d.	$d = \dfrac{16\ 000(1.007)^{60}(1.007-1)}{1.007^{60}-1}$ $= \$327.49$
3 Write a statement.	$\$16\ 000$ to be paid off in 5 years at 8.4% p.a. will need monthly repayments of $\$327.49$.
b **1** Using Finance Solver on your CAS, enter the following values: n (N:) $=$ unknown r (I (%) :) $= 8.4$ V_0 (PV:) $= 16\ 000$ d (Pmt:) $= -393.62$ V_n (FV:) $= 0$ PpY: $= 12$ CpY: $= 12$	**b** $n = 47.999\ 695\ 088\ 867$
2 Solve for n.	$n = 48$ months
3 Interpret the results.	Time $= \dfrac{48}{12}$ years $= 4$ years
4 Write a statement.	The new term of the loan would be 4 years.
c Interest paid $=$ total repayments $-$principal repaid	**c** When term $= 4$ years, $d = 393.62$. Interest $= 48 \times 393.62 - 16\ 000$ $= \$2893.76$

THINK	WRITE
d 1 (a) Review the known quantities.	**d** When term $= 5$ years, $d = 327.49$.
(b) Find the interest difference.	$\text{Interest} = 60 \times 327.49 - 16\,000$ $= \$3649.40$ $\text{Interest difference} = 3649.40 - 2893.76$ $= \$755.64$
2 Write a statement.	If the repayment is increased from \$327.49 to \$393.62 per month then \$755.64 is saved in interest payments.

If a borrower does increase the value of each repayment and if all other variables remain the same, then the term of the loan is reduced. Conversely, a decrease in the repayment value increases the term of the loan. There are two stages to the loan, each with a different repayment.

WORKED EXAMPLE 9

Brad borrowed \$22 000 to start a business and agreed to repay the loan over 10 years with quarterly instalments of \$783.22 and interest debited at 7.4% p.a. However, after 6 years of the loan Brad decided to increase the repayment value to \$879.59. Find:

a. the actual term of the loan

b. the total interest paid

c. the interest savings achieved by increasing the repayment value.

THINK	WRITE
a 1 To find A after 6 years:	**a**
(a) First identify V_0, d, n, r and R.	$V_0 = 22\,000$, $d = 783.22$ $n = 6 \times 4 = 24$ $r = \frac{7.4}{4} = 1.85$ $R = 1.0185$
(b) Find V_{24} (the balance owing after 6 years).	$V_{24} = 22\,000(1.0185)^{24} - \frac{783.22(1.0185^{24} - 1)}{1.0185 - 1}$ $= \$10\,761.83$
2 Now find the n value to reduce \$10 761.83 to zero; that is, the remaining part of the loan. Identify V_0 and d.	$V_0 = 10\,761.83$, $d = 879.59$

3 Using Finance Solver, enter the appropriate values. n (N:) = unknown r (I(%):) = 7.4 V_0 (PV:) = 10 761.9 d (Pmt:) = −879.59 V_n (FV:) = 0 PpY: = 4 CpY: = 4	$n = 14.000\ 122\ 313\ 731$
4 Solve for n.	$n = 14$ quarters
5 Interpret the results.	$\text{Time} = 3\frac{1}{2}$ years
6 Find the total term of the loan.	$\text{Total term} = 6 + 3\frac{1}{2} = 9\frac{1}{2}$ years
b For the two-repayment scenario: Interest paid = Total repayments = −Principal repaid In this case, in two stages.	**b** For the two-repayment scenario: $\text{Interest} = 783.22 \times 24 + 879.59 \times 14 - 22\ 000 = \$9111.54$
c **1** For the same repayment scenario: $d = 783.22$ for 10 years.	**c** For the same repayment scenario: $\text{Interest} = 783.22 \times 40 - 22\ 000 = \$9328.80$
2 Find the difference between the two scenarios.	$\text{Interest difference} = 9328.80 - 9111.54 = \$217.26$
3 Write a statement.	Brad will save $217.26 interest by increasing his repayment value.

○ 7.4 – Reducing Balance Loans III

WORKED EXAMPLE 10

Sharyn takes out a loan of $5500 to pay for solar heating for her pool. The loan is to be paid in full over 3 years with quarterly payments at 6% p.a.

a. Calculate the quarterly payment required.

b. Complete an amortisation table for the loan with the following headings.

Payment no.	Principal outstanding ($)	Interest due ($)	Payment ($)	Loan outstanding ($)
1	5500			

THINK	WRITE
a Using Finance Solver, enter the appropriate values. n (N:) = 12 r (I(%):) = 6 V_0 (PV:) = −5500 d (Pmt:) = unknown V_n (FV:) = 0 PpY: = 4 CpY: = 4	**a** $d = 504.239\ldots$ So the quarterly payments are $504.24.

b **1** Place $504.24 in the first payment column. Calculate the interest with the initial principal of $5500. Loan outstanding is the addition of principal and interest less the payment made. Complete the first line as shown.

b $I = 5500\left(\frac{0.06}{4}\right)$

$= 82.50$

Payment	Principal outstanding ($)	Interest due ($)	Payment ($)	Loan outstanding ($)
1	5500	82.50	504.24	5078.26

2 Complete the following: Principal outstanding = previous loan Interest due: calculated as previous with the new principal Payment: stays the same Loan outstanding = principal + interest – payment

Payment	Principal outstanding ($)	Interest due ($)	Payment ($)	Loan outstanding ($)
1	5500	82.50	504.24	5078.26
2	5078.26	76.17	504.24	4650.19

3 Complete the table following the previous steps.

Payment	Principal outstanding ($)	Interest due ($)	Payment ($)	Loan outstanding ($)
1	5500	82.50	504.24	5078.26
2	5078.26	76.17	504.24	4650.19
3	4650.19	69.75	504.24	4215.70
4	4215.70	63.24	504.24	3774.70
5	3774.70	56.62	504.24	3327.08
6	3327.08	49.91	504.24	2872.75
7	2872.75	43.09	504.24	2411.60
8	2411.60	36.17	504.24	1943.53
9	1943.53	29.15	504.24	1468.44
10	1468.44	22.03	504.24	986.23
11	986.23	14.79	504.24	496.78
12	496.78	7.45	504.24	– 0.01

- **FREQUENCY OF REPAYMENTS**

-In this section we investigate the effect on the actual term of the loan, and on the total amount of interest charged, of making more frequent repayments.

-While the value of the repayment will change, the actual outlay will not (eg. $3000 quarterly repayment compared to a $1000 monthly repayment).

-Therefore, the same amount is repaid during the same period of time in each case, the only variable is how often repayments are actually made.

-Interest will be charged just before a repayment is made, although this may not be the case in practice.

WORKED EXAMPLE 11

Tessa wants to buy a dress shop. She borrows $\$15\ 000$ at 8.5% p.a. (debited prior to each repayment) of the reducing balance. She can afford quarterly repayments of $\$928.45$ and this will pay the loan in full in exactly 5 years.

One-third of the quarterly repayment gives the equivalent monthly repayment of $\$309.48$. The equivalent fortnightly repayment is $\$142.84$.

Find:

i. the term of the loan and

ii. the amount still owing prior to the last payment if Tessa made repayments:

a. monthly

b. fortnightly.

THINK	WRITE
a i 1 Identify the given values. Enter the appropriate values using Finance Solver. Remember that PpY: $= 12$ and CpY: $= 12$ for monthly repayments.	**a i** For monthly repayments: $V_0 = 15\ 000, d = 309.48,$ $I = 8.5\%$ p.a., $n = ?$
2 Solve for n.	$n = 59.582\ 518\ 723\ 273$
3 The value obtained for n is 59.58 which means that a 60th repayment is required. That is, $n = 60$.	$n = 60$ months Term of loan $= 5$ years
ii 1 To find the amount still owing prior to the last payment, find V_n when $n = 59$. Enter the appropriate values using Finance Solver: $n = 59$ $I = 8.5$ $PV = 15\ 000$ $Pmt = -309.48$ PpY: $= 12$ CpY: $= 12$	**ii** $V_{59} = -179.273\ 603\ 537\ 66$
2 Solve for V_{59}.	
3 State the amount still owing.	The amount still owing prior to the last payment is $\$179.27$.
b i 1 Enter the appropriate values using Finance Solver on your CAS. Remember that PpY: $= 26$ and CpY: $= 26$ for fortnightly repayments.	**b i** For fortnightly repayments: $V_0 = 15\ 000, d = 142.84,$ $I = 8.5\%$ p.a., $n = ?$
2 Solve for n.	$n = 128.847\ 104\ 593\ 38$
3 The value obtained for n is 128.85 which means that a 129th repayment is required. That is, $n = 129$.	$n = 129$ fortnights Term of loan $= 4$ years, 25 fortnights

ii **1** To find the amount still owing prior to the last payment, find V_n (or FV) when $n = 128$. Enter the appropriate values using Finance Solver:

$n = 128$

$I = 8.5$

$PV = 15\ 000$

$Pmt = -142.84$

$\text{PpY:} = 26$

$\text{CpY:} = 26$

2 Solve for V_{128}.

ii $V_{128} = -120.636\ 212\ 821\ 95$

3 State the amount still owing.

The amount still owing prior to the last payment is $\$120.64$.

It can be seen from Worked example 11 that while the same outlay is maintained there may be a slight decrease in the term of a loan when repayments are made more often. Let us now find what the saving is for such a loan. In this situation we should consider the final (partial) payment separately because the amount of interest that it attracts is less than a complete repayment, d.

The calculation of the total interest paid is now calculated as usual:

$$\text{Total interest} = \text{total repayments} - \text{principal repaid.}$$

WORKED EXAMPLE 12

In Worked example 11, Tessa's $\$15\ 000$ loan at 8.5% p.a. gave the following three scenarios:

1. **quarterly repayments of $\$928.45$ for 5 years**
2. **monthly repayments of $\$309.48$ for 59 months with $\$179.27$ still outstanding**
3. **fortnightly repayments of $\$142.84$ for 128 fortnights with $\$120.64$ still owing.**

Compare the total interest paid by Tessa if she repaid her loan:

a. **quarterly**

b. **monthly**

c. **fortnightly.**

THINK

a For quarterly repayments Total interest = total repayments − principal repaid

WRITE

a For quarterly repayments:

$$\begin{aligned}\text{total interest} &= 928.45 \times 20 - 15\ 000\\ &= \$3569\end{aligned}$$

b **1** For monthly repayments Find r and V_{59}. (Refer to Worked example 11.)

b For monthly repayments:

$$r = \frac{8.5}{12}\% = 0.7083\%$$

$$V_{59} = \$179.27$$

2 Calculate the interest on V_{59} to find the final repayment.

$$\begin{aligned}\text{Interest on } V_{59} &= 0.7083\% \text{ of } \$179.27\\ &= 0.007\ 083 \times 179.27\\ &= \$1.27\end{aligned}$$

$$\begin{aligned}\text{Final repayment} &= 179.27 + 1.27\\ &= \$180.54\end{aligned}$$

3 Calculate the total interest paid.

$$\begin{aligned}\text{Total interest} &= 309.48 \times 59 + 180.54 - 15\ 000\\ &= \$3439.86\end{aligned}$$

c 1	Fortnightly repayments Find r and V_{128}. (Refer to Worked example 11.)	**c** For fortnightly repayments: $r = \frac{8.5}{26}\% = 0.3269\%$ $V_{128} = 120.64$
2	Calculate the interest on V_{128} to find the final repayment.	Interest on V_{128} = 0.3269% of \$120.64 $= 0.003\ 269 \times 120.64$ $= \$0.39$ Final repayment $= 120.64 + 0.39$ $= \$121.03$
3	Calculate the total interest paid.	Total interest $= 142.84 \times 128 + 121.03 - 15\ 000$ $= \$3404.55$
4	Calculate the interest saving with monthly repayments over quarterly repayments.	Monthly interest saving $= 3569 - 3439.86$ $= \$129.14$
5	Calculate the interest saving with fortnightly repayments over quarterly repayments.	Fortnightly interest saving $= 3569 - 3404.55$ $= \$164.45$
6	Write a comparison statement.	Tessa saves \$164.45 if she repays fortnightly rather than quarterly and \$129.14 if she repays monthly rather than quarterly.

-In this section, we investigate the effect that changing the interest rate has on the term of the loan and on the total interest paid.

First, let us simply compare loan situations by varying only the rate.

WORKED EXAMPLE 13

A reducing balance loan of \$18 000 has been taken out over 5 years at 8% p.a. (adjusted monthly) with monthly repayments of \$364. 98.

a. What is the total interest paid?

b. If, instead, the rate was 9% p.a. (adjusted monthly) and the repayments remained the same, what would be:

i. the term of the loan

ii. the total amount of interest paid?

THINK	WRITE
a For 8% p.a.: Total interest = total repayments – principal repaid	**a** For 8% p.a.: Total interest $= 364.98 \times 60 - 18\ 000$ $= \$3898.80$
b i Using the Finance Solver, enter the appropriate values. Remember that PpY: and CpY: will both equal 12 for monthly repayments. N: = 61.81 means 61 full repayments plus a final lesser payment.	**b i** For 9% p.a.: $V_0 = 18\ 000,\ d = 364.98,\ I = 9,\ V_n = 0,$ $n = ?$ $n = 61.810\ 665\ 384\ 123$ $n = 62$ months Term = 5 years, 2 months

ii 1 Find V_{61} to calculate the amount still owing. Enter the following values: $n = 61$, $I = 9$, $PV = 18\ 000$, $Pmt = -364.98$, PpY: $= 12$, CpY: $= 12$. Solve for FV and interpret the result.	**ii** FV $= -293.886672877$ The amount still owing after 61 repayments is $293.88.
2 Calculate the interest on V_{61} to find the final repayment. $r = \frac{9}{12} = 0.75\%$	Interest on final repayment $= 0.75\%$ of $293.8 $= \$2.20$ Final repayment $= 293.88 + 2.20$ $= \$296.08$
3 Total interest = total repayments – principal repaid.	Total interest paid $= 364.98 \times 61 + 296.08 - 18\ 000$ $= \$4559.86$

-The rate was increase by only 1% p.a. on $18000 for only 5 years, yet the amount of interest paid has increased from $3898.80 to $4559.86, a difference of $661. 06.

-This difference takes on even more significant proportions over a longer period of time and with a larger Vo.

WORKED EXAMPLE 14

Natsuko and Hymie took out a loan for home renovations. The loan of $42 000 was due to run for 10 years and attract interest at 7% p.a., debited quarterly on the outstanding balance. Repayments of $1468. 83 were made each quarter. After 4 years the rate changed to 8% p.a. (debited quarterly). The repayment value didn't change.

a. Find the amount outstanding when the rate changed.

b. Find the actual term of the loan.

c. Compare the total interest paid to what it would have been if the rate had remained at 7% p.a. for the 10 years.

THINK	WRITE
a 1 Rate changes after 4 years; that is, $n = 16$. State V_0 (PV:), d (Pmt:), interest (I(%):) and n (N:).	**a** $V_0 = 42\ 000,\ d = 1468.83,\ I = 7\%,$ $n = 16, V_n = ?$
2 Find V_{16} (FV:) using a CAS calculator.	FV $= -285\ 84.356\ 811\ 602$ The amount outstanding when the rate changed is $28 584.36.

b **1** Find n to repay $\$28\ 584.36$ in full at the new rate. (Use Finance Solver to enter the following values: $I = 8$, $PV = 28\ 584.36$, $Pmt = -1468.83$, $\text{FV} = 0$, $\text{PpY:} = 4$ and $\text{CpY:} = 4$)

b New interest rate of 8%:

$$n = -24.896\ 005\ 939\ 422$$
$$= 25 \text{ quarters}$$
$$\text{Time} = 6\frac{1}{4} \text{ years}$$

2 Find the total term of the loan, that is, time at 7% plus time at 8%.

$$\text{Term} = 4 \text{ years} + 6\frac{1}{4} \text{ years}$$
$$= 10\frac{1}{4} \text{ years}$$

c **1** Find V_{24} to calculate the amount still owing. (That is, on your calculator change the value of n to 24 and solve for FV.)

c $\text{FV} = -1291.669\ 456\ 564\ 649$

The amount still owing after the 24th repayment is $\$1291.61$.

2 Calculate the interest on the outstanding amount to find the final repayment. $r = \frac{8}{4} = 2\%$

$$\text{Interest on final payment} = 2\% \text{ of } \$1291.61$$
$$= \$25.83$$
$$\text{Final repayment} = 1291.61 + 25.83$$
$$= \$1317.44$$

3 Find the total interest for the rate change scenario. The number of repayments is 40 at $\$1468.83$ plus 1 at $\$1317.44$.

For the rate change scenario, total interest

$$= 1468.83 \times 40 + 1317.44 - 42\ 000$$
$$= \$18\ 070.64$$

4 Calculate the total interest if the rate remained at 7%.

For the rate at 7% only, total interest

$$= 1468.83 \times 40 - 42\ 000$$
$$= \$16\ 753.20$$

5 Find the interest difference between the two scenarios.

$$\text{Interest difference} = 18\ 070.64 - 16\ 753.20$$
$$= \$1317.44$$

6 Write a comparison statement.

An extra $\$1317.44$ interest will be paid due to the interest rate change from 7% p.a. to 8% p.a.

Interest only loans

Interest only loans are loans where the borrower makes only the minimum repayment equal to the interest charged on the loan. As the initial principal and amount owing is the same for the period of this loan, we could use either the simple interest formula or CAS to solve problems of this type. When using Finance Solver, the present value (PV:) and future value (FV:) are entered as the same amount. Note that the future value is negative to indicate money owed to the bank.

This type of loan is used by two kinds of borrowers: investors in shares and/or property or families that are experiencing financial difficulties and seek short-term relief from high repayment schedules.

WORKED EXAMPLE 15

Jade wishes to borrow $\$40\ 000$ to invest in shares. She uses an interest only loan to minimise her repayments and hopes to realise a capital gain when she sells the shares at a higher value. The term of the loan is 6.9% p.a. compounded monthly with monthly repayments equal to the interest charged.

a. Calculate the monthly interest-only repayment.

b. If, in 3 years, she sells the shares for $\$50\ 000$, calculate the profit she would make on this investment strategy.

THINK		WRITE
a	1 Identify V_0, r and n, where n is equal to one payment period.	a $V_0 = 40\,000,\ r = 6.9\%$ p.a. $n = 1$ month $= \frac{1}{12}$th year
	2 Evaluate I using the simple interest formula.	$I = \frac{V_0 rn}{100}$ $= \frac{40\,000 \times 6.9 \times \frac{1}{12}}{100}$ $= \$230$
	3 Write your answer.	The monthly repayment to pay the interest only for the loan is $230.
b	1 Find whether the capital gain on the shares exceeds the amount paid in interest.	b Capital gain = selling price − purchase price = $50 000 − $40 000 = $10 000 Total interest charged = repayment × number of payments = $230 × 36 = $8280 Profit = Capital gain − Loan cost = $10 000 − $8280 = $1720
	2 Write a statement.	Jade will make a profit of $1720.

○ 7.5 – Reducing Balance Loans and Flat Rate Loan Comparisons

As we have seen in previous sections, with reducing balance loans, interest is calculated on the current balance and debited to the loan account at regular intervals just before repayments are made. Since the balance continually reduces, the amount of interest charged also reduces.

In contrast, **flat rate loans** charge a fixed amount of interest as a percentage of the original amount borrowed. This is calculated at the start of a loan and added to the amount borrowed. Since it is a flat rate based on a fixed amount, the simple interest formula is used to calculate the interest:

$$I = \frac{V_0\ rn}{100}.$$

Let us compare the two types of loan under similar circumstances.

WORKED EXAMPLE 16

A loan of $\$12\ 000$ is taken out over 5 years at 12% p.a. Find:

a. the monthly repayment

b. the total amount of interest paid

if the money is borrowed on:

i. a flat rate loan

ii. a reducing balance loan.

THINK	WRITE
a i 1 For a flat rate loan: State V_0, r and n. Find the interest by using $I = \frac{V_0 rn}{100}$.	**a i** For a flat rate loan: $V_0 = 12\ 000,\ r = 12,\ n = 5$ $I = \frac{12\ 000 \times 12 \times 5}{100}$ $= \$7200$
2 Find the amount to repay, V_n. Total repaid $= V_0 + I$.	$V_n = V_0 + I$ $= 12\ 000 + 7200$ $= \$19\ 200$
3 Find the monthly repayment. (First find n.)	$n = 12 \times 5$ $= 60$ $\text{Repayment} = 19\ 200 \div 60$ $= \$320/\text{month}$
b i State the total amount of interest paid.	**b i** The total amount of interest paid is $\$7200$.
a ii For a reducing balance loan: (a) Find V_0, n, r and R.	**a ii** For a reducing balance loan: $V_0 = 12\ 000,\ n = 5 \times 12 = 60$ $r = \frac{12}{12}$ $= 1$ $R = 1.01$
(b) Find the monthly repayment, d, using $d = \frac{V_0 R^n (R-1)}{R^n - 1}$.	$d = \frac{12000(1.01)^{60}(1.01-1)}{1.01^{60}-1}$ $= \$266.93/\text{month}$ The monthly repayment is $\$266.93$.
b ii Find the total interest paid (total repayments − principal repaid).	**b ii** Total interest paid $= 266.93 \times 60 - 12\ 000$ $= \$4015.80$

In Worked example 16, the difference between the two loan types is significant. For the reducing balance loan, each month $\$53.07$ less is repaid and overall $\$3184.20$ less interest is paid.

The percentage saving over this short loan is:

$$\text{percentage interest saving} = \frac{3184.20}{7200} \times 100\%$$
$$= 44.23\%.$$

Choosing a reducing balance loan rather than a flat rate loan results in a smaller repayment value or a shorter term and in both cases an interest saving. Now let us consider what flat rate of interest is equivalent to the rate for a reducing balance loan.

WORKED EXAMPLE 17

A reducing balance loan of $\$25\ 000$ is repaid over 8 years with monthly instalments and interest charged at 9% p.a. (debited monthly).

Find:

a. **the repayment value**

b. **the total amount of interest paid**

c. **the equivalent flat rate of interest for a loan in which all other variables are the same.**

THINK	WRITE
a For a reducing balance loan: V_0, n, r and R.	**a** For a reducing balance loan: $V_0 = 25\ 000$, $n = 8 \times 12 = 96$. $r = \frac{9}{12} = 0.75$ $R = 1.0075$
Find d.	$d = \frac{V_0 R^n (R-1)}{R^n - 1} = \frac{25\ 000(1.0075)^{96}(1.0075-1)}{(1.0075^{96}-1)} = \$366.26$
b Total interest = total repayments − principal rpaid	**b** Total interest $= 366.26 \times 96 - 25\ 000 = \$10\ 160.96$
c For a flat rate loan: (i) State I, V_0 and n.	**c** For a flat rate loan: I = interest from reducing balance loan $= \$10\ 160.96$ $V_0 = 25\ 000$, $n = 8$
(ii) Find r using $I = \frac{V_0 r n}{100}$.	$10\ 160.96 = \frac{25\ 000 \times r \times 8}{100} = 2000 \times r$ $r = 5.08\%$ The equivalent flat rate of interest is 5.08%.

-Worked Example 17 illustrates that an interest rate of 9% p.a. on the outstanding balance is equivalent to a **flat rate** of only 5.08% p.a., which again is a major difference between the two loan types.

Finally, we consider the effect on the amount that can be borrowed at a given rate for both types of loan.

WORKED EXAMPLE 18

A loan of $\$76\,000$ is repaid over 20 years by quarterly instalments of $\$2205.98$ and interest is charged quarterly at 10% p.a. of the outstanding balance.

Find:

a. the total amount of interest paid

b. the amount which can be borrowed on a flat rate loan in which all other variables are the same as above

c. the difference in the amount borrowed between the two types of loan.

THINK	WRITE
a 1 For a reducing balance loan: Find the total interest.	**a** For a reducing balance loan: $n = 20 \times 4 = 80$; Interest $= 2205.98 \times 80 - 76\,000 = \$100478.40$
2 Write a statement.	The total amount of interest paid is $\$100\,478.40$.
b 1 For a flat rate loan: (a) State I, r and n. I is the same as for the reducing balance loan. (b) Use the simple interest formula to find V_0.	**b** For a flat rate loan: $I = 100\,478.40,\ r = 10,\ n = 20$; $I = \frac{V_0 r\, n}{100}$; $100\,478.40 = \frac{V_0 \times 10 \times 20}{100} = 2 \times V_0$; $V_0 = \$50\,239.20$
2 Write a statement.	For a flat rate loan, $\$50\,239.20$ can be borrowed.
c 1 Find the difference between the principals for the two loan types.	**c** The difference in the amount borrowed $= 76\,000 - 50\,239.20 = \$25\,760.80$
2 Write a statement.	Under the same conditions a $\$76\,000$ reducing balance loan is equivalent to a $\$50\,239.20$ flat rate loan.

-The greater financial benefit of the **reducing balance** loan over the **flat rate** loan is again evident, this time in terms of the amount that can be borrowed in the first place, V_0.

○ 7.6 – Effective Annual Interest Rate, r_e

Previously we have looked at paying off a loan at a set interest rate, however we have found the amount of interest paid would vary with different compounding terms (daily, weekly, monthly, etc.). The **effective annual interest rate** is used to compare the annual interest between loans with these different compounding terms.

To calculate the effective annual interest rate, use the formula:

$$r = \left(1 + \frac{i}{n}\right)^n - 1$$

where

r **= the effective annual interest rate**

i **= the nominal rate, as a decimal**

n **= the number of compounding periods per year**

WORKED EXAMPLE 19

Jason decides to borrow money for a holiday. If a personal loan is taken over 4 years **with equal quarterly repayments compounding at** 12% p.a.**, calculate the effective annual rate of interest (correct to** 2 decimal **places).**

THINK	WRITE
1 Write the values for i and n.	$n = 4$ (since quarterly) $i = 0.12$ (12% as a decimal)
2 Write the formula for effective annual rate of interest.	$r = \left(1 + \frac{i}{n}\right)^n - 1$
3 Substitute $n = 4$ and $i = 0.12$.	$r = \left(1 + \frac{0.12}{4}\right)^4 - 1$ $= 0.1255$ $= 12.55\%$
4 Write your answer.	The effective annual interest rate is 12.55% p.a. for a loan of 12%, correct to 2 decimal places.

○ 7.7 – Perpetuities

A **perpetuity** is an annuity where a permanently invested sum of money provides regular payments that continue forever.

Many *scholarships* or *grants* offered to students at universities are provided by funds known as perpetuities.

The funds last for an indefinite period of time as long as the amount paid out is no more than the interest earned on the initial lump sum deposited. The type of investment that is used to earn the interest is usually a bond, which offers a fixed interest amount, paid on a regular basis, over a long period of time. Wealthy people who wish to encourage and support a worthwhile cause usually set up these perpetuities.

The balance of the amount invested does not change and is the same for an indefinite period.

The perpetuity formula is:

$$d = \frac{V_0 r}{100}$$

where

d = **the amount of the regular payment per period** $(\$)$

V_0 = **the principal** $(\$)$

r = **the interest rate earned per period (%).**

Notes

1. The period of the regular payment must be the same as the period of the given interest rate.
2. Finance Solver can be used in calculations involving perpetuities. As the principal does not change, the present value (PV: or negative cash flow) and the future value (FV: or positive cash flow) are entered as the same amount, but with opposite signs.

WORKED EXAMPLE 20

Robert wishes to use part of his wealth to set up a scholarship fund to help young students from his town further their education at university. Robert invests $\$200\ 000$ in a bond that offers a long-term guaranteed interest rate of 4% p.a. If the interest is calculated once a year, then the annual amount provided as scholarship will be:

A. $\$188\ 000$

B. $\$288\ 000$

C. $\$666.67$

D. $\$8000$

E. $\$4000$

THINK	WRITE
1 Write the perpetuity formula.	$d = \frac{V_0 r}{100}$
2 List the values of V_0 and r.	$V_0 = \$200\ 000$ and $r = 4\%$ p.a.
3 Substitute the values into the formula and calculate the amount provided.	$d = \frac{\$200\ 000 \times 4}{100}$ $= 8000$
4 Select the appropriate answer.	The annual amount provided for the scholarship is $\$8000$. Therefore D is the correct answer.

Finding V_0 and r

As was the case with earlier sections in this topic, there are calculations where we need to find the principal (V_0) or interest rate (r) needed to provide a certain regular payment (d). For example, how much needs to be invested at 3% p.a. interest to provide a $\$10\ 000$ annual grant, or what interest rate is needed so that $\$100\ 000$ will provide a $\$4000$ yearly scholarship indefinitely? Other calculations involve finding what extra amount could be granted annually as a scholarship if the interest is compounded monthly in each year rather than once a year and the scholarship paid in two equal six-monthly instalments.

The perpetuity formula can be transposed to:

$$V_0 = \frac{100 \times d}{r} \text{ and } r = \frac{100 \times d}{V_0}$$

If the frequency of the payments each year is not the same as the compounding period of the given interest rate, then Finance Solver is to be used with different values for PpY and CpY.

Notes:

1. The principal must be known to use Finance Solver.
2. Finance Solver gives the interest rate per annum.

WORKED EXAMPLE 21

A Rotary Club has $\$100\ 000$ to set up a perpetuity as a grant for the local junior sporting clubs. The club invests in bonds that return 5.2% p.a. compounded annually.

a. Find the amount of the annual grant.

b. What interest rate (compounded annually) would be required if the perpetuity is to provide $\$6000$ each year?

The Rotary Club wants to investigate other possible arrangements for the structure of the grant.

c. How much extra would the annual grant amount to if the original interest rate was compounded monthly?

d. What interest rate (compounded monthly) would be required to provide 4 equal payments of $\$1500$ every 3 months? Give your answer correct to 2 decimal places.

THINK	WRITE
a 1 Write the perpetuity formula and list the values of V_0 and r.	**a** $d = \frac{V_0 r}{100}$ $V_0 = \$100\ 000$ and $r = 5.2$, p. a.
2 Substitute the values into the formula and find the value of the annual grant.	$d = \frac{\$100\ 000 \times 5.2}{100}$ $= 5200$
3 Write a statement.	The amount of the annual grant is $\$5200$.
b 1 Write the perpetuity formula and list the values of V_0 and d.	**b** $r = \frac{100 \times d}{V_0}$ $V_0 = \$100\ 000$ and $d = \$6000$
2 Substitute the values into the formula and find the interest rate.	$r = \frac{100 \times 6000}{100\ 000}$ $= 6$
3 Write a statement.	For a $\$100\ 000$ perpetuity to provide $\$6000$ a year, the bond needs to offer an interest rate of 6% p.a.
c 1 As the frequency of the payment is not the same as the compounding period, the perpetuity formula cannot be used. Use Finance Solver and enter the values as follows. $n(\text{N:}) = 1$ $r(\text{I}(\%)\text{:}) = 5.2$ $V_0(\text{PV:}) = -100\ 000$ $d(\text{Pmt:}) = \text{unknown}$ $V_n(\text{FV:}) = 100\ 000$ PpY: $= 1$ (one payment per year) CpY: $= 12$ (there are 12 compound periods per year) Solve for Pmt.	**c** $\text{Pmt} = 5325.741\ 057\ 054$ If the interest was compounded monthly, the annual grant would amount to $\$5325.74$.

2 Compare the sizes of the 2 grants and write a statement.

The extra amount is $\$5325.74 - \$5200 = \$125.74$. If the interest is compounded monthly, the annual grant would increase by \$125.74.

d 1 As the frequency of the payment is not the same as the compounding period, the perpetuity formula cannot be used. Use Finance Solver and enter the values as follows.

n (N:) = 1

r (I(%):) = unknown

V_0 (PV:) = −100 000

d (Pmt:) = 1500

V_n (FV:) = 100 000

PpY: = 4 (four payments per year)

CpY: = 12 (there are 12 compound periods per year)

Solve for I to find the required interest rate.

d $I = 5.970\ 247\ 527\ 183$

$r = 5.97$

2 Write a statement.

An interest rate of 5.97% p.a. compounded annually is needed to provide four equal payments of \$1500, correct to 2 decimal places.

WORKED EXAMPLE 22

A benefactor of a college has been approached to provide a Year 7 scholarship of \$1000 per term. He is able to get a financial institution to offer a long-term interest rate of 8% per annum. What is the principal that needs to be invested?

THINK	WRITE
1 Write the perpetuity formula and list the values of d and r. Both d and r need to be expressed in the same period of time.	$V_0 = \frac{100 \times d}{r}$ $d = \$1000$ per term (4 terms per year) $R = 8\%$ p.a. $= \frac{8}{4}$ $= 2\%$ per term
2 Substitute the values into the formula and find the value of the annual grant.	$V_0 = \frac{100 \times 1000}{2}$ $= 50\ 000$
3 Write a statement.	The principal that needs to be invested to provide a scholarship of \$1000 per term at an annual interest rate of 8% is \$50 000.

Note that Finance Solver cannot be used in the previous worked example as the principal is not known. (Both PV: and FV: would be unknowns.)

○ *7.8 – Annuity Investments*

-An option with a managed fund (where you invest an initial V_0 and rely on the fund managers to invest wisely and hope on a +ve % return) is to contribute to the fund, increasing the V_0 and thus increasing the interest earned.

-This is another example of a **first-order recurrence relation**.

WORKED EXAMPLE 23

Johnathan invested $5000 in a managed fund that will earn an average of 8% p.a. over a 2 year period with interest calculated monthly. If Johnathan contributes $100 at the start of the second, third, fourth and fifth months, complete the table to find the value of his investment at the end of the fifth month.

Time period	Principal ($)	Interest earned ($)	Balance ($)
1	5000		
2			
3			
4			
5			

THINK

WRITE

1 Calculate the interest earned in the first month, allowing for the monthly interest.

$$I = 5000\left(\frac{0.08}{12}\right)$$
$$= 33.33$$

2 Calculate the balance by adding the initial principal and the interest earned.

$$\text{Balance} = 5000 + 33.33$$
$$= 5033.33$$

3 The new principal at the start of the second period is the sum of the balance at the end of the first period and the $100 added.

$$\text{Principal} = 5033.33 + 100$$
$$= 5133.33$$

4 Repeat the above steps to complete the table.

Time period	Principal ($)	Interest earned ($)	Balance ($)
1	5000	33.33	5033.33
2	5133.33	34.22	5167.55
3	5267.55	35.12	5302.67
4	5402.67	36.02	5438.69
5	5538.69	36.92	5575.61

5 Write the answer to the value of the investment after 5 months.

After five months the investment is worth $5575.61.

-A savings plan is an investment where an initial sum as well as regular deposits are made.

-The interest earned is calculated regularly on the balance of the investment, which increases with each regular deposit (annuity).

-This is therefore similar to **reducing balance loans** with the main difference being that the V_0 is *growing*.

An **annuity investment** is an investment that has regular deposits made over a period of time. Let's first look at this as a recurrence relation that calculates the value of the annuity after each time period.

$$V_{n+1} = V_n R + d$$
$$= V_n\left(1 + \frac{r}{100}\right) + d$$

Where: V_{n+1} = **Amount after** $n+1$ **payments**

V_n = **Amount at time** n

r = **Interest rate per period**

d = **Deposit amount**

WORKED EXAMPLE 24

An initial deposit of $\$1000$ was made on an investment taken out over 5 years at a rate of 5.04% p.a. (interest calculated monthly), and an additional deposit of $\$100$ is made each month. Complete the table below for the first five deposits and calculate how much interest had been earned over this time.

$n+1$	V_n	d	V_{n+1}
1	$1000		
2			
3			
4			
5			

THINK

1 State the initial values for V_0, r and d.

WRITE

$V_0 = \$1000$

$r = \frac{5.04}{12} = 0.42$

$d = \$100$

THINK

2 Evaluate V_1.

WRITE

$$V_{n+1} = V_n\left(1 + \frac{r}{100}\right) + d$$
$$V_1 = 100\left(1 + \frac{0.42}{100}\right) + 100$$
$$= \$1104.20$$

3 Evaluate V_2.

$$V_{n+1} = V_n\left(1+\frac{r}{100}\right)+d$$
$$V_2 = 1104.20\left(1+\frac{0.42}{100}\right)+100$$
$$= \$1208.84$$

4 Evaluate V_3.

$$V_{n+1} = V_n\left(1+\frac{r}{100}\right)+d$$
$$V_3 = 1208.84\left(1+\frac{0.42}{100}\right)+100$$
$$= \$1313.92$$

5 Evaluate V_4.

$$V_{n+1} = V_n\left(1+\frac{r}{100}\right)+d$$
$$V_4 = 1313.92\left(1+\frac{0.42}{100}\right)+100$$
$$= \$1419.44$$

6 Evaluate V_5.

$$V_{n+1} = V_n\left(1+\frac{r}{100}\right)+d$$
$$V_5 = 1419.44\left(1+\frac{0.42}{100}\right)+100$$
$$= \$1525.40$$

7 Complete the table.

$n+1$	V_n	d	V_{n+1}
1	$V_0 = \$1000$	$\$100$	$V_1 = \$1104.20$
2	$V_1 = \$1104.20$	$\$100$	$V_2 = \$1208.84$
3	$V_2 = \$1208.84$	$\$100$	$V_3 = \$1313.92$
4	$V_3 = \$1313.92$	$\$100$	$V_4 = \$1419.44$
5	$V_4 = \$1419.44$	$\$100$	$V_5 = \$1525.40$

8 Calculate the interest by subtracting the initial investment ($\$1000$) and the amount deposited $(5 \times \$100)$ from the amount at the end of the 5 deposits.

$$I = \$1525.40 - \$1000 - \$500$$
$$= \$25.40$$

9 State the answer.

The interest earned over the first five months was $\$25.40$.

Superannuation

Most, if not all, Australians will have to provide for themselves in their retirement rather than relying on the government's age pensions. To provide for their future, all working Australians have a **superannuation** fund into which money is contributed by their employers, and optionally topped up by the employee, each pay period. The sum accumulates over many years, until retirement age, when the money can be withdrawn. The funds can then be placed into an annuity or perpetuity that pays for the retiree's living expenses and lifestyle.

This is where superannuation calculations become tricky. You need to work out the amount of money you must save to give you that retirement income. This depends on all sorts of variables, such as the initial deposit, regular instalments, investment returns, inflation and tax rates.

The accumulated money, deposited by the workers, is invested in shares and properties, over many years by financial institutions (also known as superannuation fund managers). The performance of these superannuation fund managers varies from year to year. For the scope of this exercise, we will assume a constant rate of return (interest rates remain the same). Also, the effects of inflation and taxation will not be considered.

The money that accumulates in these annuity investments can be calculated using the annuities formula in a similar way to that used in reducing balance loans.

The difference is that the amount (V_n) *grows with the addition of a regular payment* (d). The formula is:

$$V_n = V_0 R^n + \frac{d(R^n - 1)}{R - 1}$$

where:

V_0 = **the initial amount invested**

R = **the compounding or growth factor**

$= 1 + \frac{r}{100}$ (r = **the interest rate per payment period)**

d = **the amount of the regular deposits made per period**

n = **the number of deposits**

V_n = **the balance after n deposits.**

Finance Solver can also be used in a similar way to reducing balance loans, with one difference — the cash flows are reversed (opposite signs).

WORKED EXAMPLE 25

Helen currently has $\$2000$ in a savings account that is averaging an interest rate of 8% p.a. compounded annually. She wants to calculate the amount that she will receive in 5 years when she plans to go on an overseas trip.

a. **If she deposits $\$6000$ each year, find (correct to the nearest $\$1000$) the amount available for her overseas trip.**

b. **If she places her $\$2000$ and increases her deposits to $\$7000$ each year into a different savings account that can offer 9% p.a. compounded annually, find (correct to the nearest $\$1000$) the amount available for her overseas trip.**

c. **Calculate the extra amount saved by investing $\$7000$ each year at 9% p.a. compared with $\$6000$ each year at 8% p.a.**

THINK	WRITE
a **1** State the value of V_0, d, n, r and R.	**a** $V_0 = \$2000$, $d = \$6000$, $n = 5$, $r = 8$ and $R = 1.08$

2 Write the annuities formula and substitute in the values.

$$V_n = V_0 R^n + \frac{d(R^n - 1)}{R - 1}$$

$$V_5 = 2000 \times 1.08^5 + \frac{6000(1.08^5 - 1)}{1.08 - 1}$$

3 Evaluate V_5.

$= \$38\ 138.26$

4 Write a statement, rounding the answer correctly.

The final balance of the investment after 5 years is $38 000, correct to the nearest $1000.

b 1 State the value of V_0, d, n, r and R.

b $V_0 = \$2000$,
$d = \$7000$,
$n = 5$,
$r = 9$ and
$R = 1.09$

2 Write the annuities formula and substitute in the values.

$$V_n = PR^n + \frac{d(R^n - 1)}{R - 1}$$

$$V_5 = 2000 \times 1.09^5 + \frac{7000(1.09^5 - 1)}{1.09 - 1}$$

3 Evaluate V_5.

$= \$44\ 970.22$

4 Write a statement, rounding the answer correctly.

The final balance of the investment after 5 years, if Helen deposits $7000 each year into an account offering 9% p.a., would be $45 000, correct to the nearest $1000.

c The extra amount saved is the difference between the amounts found in parts **a** and **b**.

c The extra amount is $\$45\ 000 - \$38\ 000 = \$7000$.

○ ***PLANNING FOR RETIREMENT***

Planning for retirement is an issue that you'll need to revise regularly, maybe with a financial planner. The annuities formula and Finance Solver can be used to calculate how much money is needed under different financial situations.

WORKED EXAMPLE 26

Andrew is aged 45 and is planning to retire at 65 years of age. He estimates that he needs $480 000 to provide for his retirement. His current superannuation fund has a balance of $60 000 and is delivering 7% p.a. compounded monthly.

a. Find the monthly contributions needed to meet the retirement lump sum target.

b. If in the final ten years before retirement, Andrew doubles his monthly contribution calculated from a, find the new lump sum amount available for retirement.

c. How much extra could Andrew expect if the interest rate from part b is increased to 9% p.a. (for the final 10 years) compounded monthly? Round the answer correct to the nearest $1000.

THINK | **WRITE**

a 1 Identify the initial amount (V_0) and the final amount (V_n).

a $V_0 = \$60\ 000$, $V_n = \$480\ 000$

2	Find the number of payments, n, the interest rate per month, r, and the growth factor, R.	$n = 20 \times 12 = 240$ $r = 7\%$ per annum $= \frac{7}{12}\%$ per annum $= 0.58\dot{3}$ $R = 1 + \frac{r}{100}$ $= 1.0058\dot{3}$
3	Write the annuities formula and substitute in the values.	$V_n = V_0 R^n + \frac{d(R^n - 1)}{R - 1}$ $480\,000 = 60\,000 \times 1.005\,8\dot{3}^{240} + \frac{d(1.005\,8\dot{3}^{240} - 1)}{(1.005\,8\dot{3} - 1)}$ $480\,000 = 242\,324.33 + \frac{d \times 3.038\,738\,849}{0.005\,8\dot{3}}$ $= 242\,324.33 + d \times 520.93$
4	Evaluate d.	$d = \$456.26$
5	Write a statement.	The monthly contribution to achieve a retirement lump sum of $480 000 is $456.26.

b	**1**	We need to find the balance after the first ten years with $d = \$456.26$ and $n = 120$. Enter the values into the formula and solve for V_{120}.	**b** $V_n = V_0 R^n + \frac{d(R^n - 1)}{R - 1}$ $V_{120} = 60\,000 \times 1.005\,8\dot{3}^{120} + \frac{456.26 \times (1.005\,8\dot{3}^{120} - 1)}{1.005\,8\dot{3} - 1}$ $= \$199\,551.36$
	2	State the values used for the final ten years and substitute them into the formula.	$V_0 = \$199\,551.36, n = 120, R = 1.0058\dot{3}$ $d = \$912.52 \quad (2 \times 456.26)$ $V_n = V_0 R^n + \frac{d(R^n - 1)}{R - 1}$ $V_{120} = 199\,551.36 \times 1.005\,8\dot{3}$ $+ \frac{912.52 \times (1.005\,8\dot{3}^{120} - 1)}{1.005\,8\dot{3} - 1}$
	3	State the new value of V_{120}.	$= \$558\,974.01$
	4	Write a statement.	The new lump sum will be $558 974.00.

c **1** Calculate the new growth factor.	**c**	$r = 9\%$ per annum $= \frac{9}{12}\%$ per month $= 0.75$ $R = 1 + \frac{r}{100}$ $= 1.0075$
2 Substitute the values into the formula for V_n and evaluate.		$V_n = V_0 R^n + \frac{d(R^n - 1)}{R - 1}$ $V_{120} = 199\,551.36 \times 1.0075^{120} + \frac{912.52 \times (1.0075^{120}) - 1}{1.0075 - 1}$ $= \$665\,757.29$
3 Find how much extra is expected by finding the difference between the two amounts.		The difference expected is $\$665\,757.29 - \$558\,974.01 = \$106\,783.28$.
4 Write a statement, rounding the answer appropriately.		If the interest rate is increased to 9% for the final 10 years, Andrew could expect an extra $\$107\,000$, correct to the nearest $\$1000$.

Once a lump sum has been realised, the funds are transferred or rolled over to a suitable annuity. This annuity will then provide a regular income to live on. There are two options:

1. Perpetuities. As seen in the previous exercise, these annuities provide a regular payment forever. This has two benefits. Firstly, it will provide for the retiree no matter how long they live and secondly, the perpetuity could be willed to relatives who in turn will collect the same annuity indefinitely.
2. Annuity — reducing balance. This is the same as reducing balance loans except the fund manager borrows the money and pays the retiree a regular income for a specified number of years. The main disadvantage is if the retiree outlives the term of the reducing balance annuity; that is, the money runs out.

WORKED EXAMPLE 27

Jarrod is aged 50 and is planning to retire at 55. His annual salary is $70 000 and his employer contributions are 9% of his gross monthly income. Jarrod also contributes a further $500 a month as a salary sacrifice (that is, he pays $500 from his salary into the superannuation fund). The superfund has been returning an interest rate of 7.2% p.a. compounded monthly and his current balance in the superfund is $255 000.

a. Calculate Jarrod's total monthly contribution to the superannuation fund.

b. Calculate the lump sum that he can receive for his planned retirement at age 55.

Jarrod has two options for setting up an annuity to provide a regular income after he retires at 55.

1. A perpetuity that offers monthly payments at 8% p.a. compounded monthly.

2. A reducing balance annuity, also paid monthly at 8% p.a., compounded monthly.

c. Calculate the monthly annuity using option 1. Express the annual salary from this option as a percentage of his current salary.

d. Calculate the monthly annuity using option 2 if the fund needs to last for 25 years. Express the annual salary from this option as a percentage of his current salary.

THINK	WRITE
a 1 Calculate the total contributions made by Jarrod's employer.	**a** The employer contribution is 9% of the gross monthly income. 9% of $\dfrac{70\ 000}{12}$ $= 0.09 \times 5833.33$ $= \$525$

	2 Calculate the total monthly contribution made to the superannuation fund.	The total monthly contributions are the employer contributions and Jarrod's contribution. That is, $\$525 + \$500 = \$1025$.
b	**1** State the value of V_0, d, n and R.	**b** $V_0 = \$255\,000, d = \$1025, n = 60, R = 1.006$
	2 Write the annuities formula and substitute in the values of the pronumerals. Evaluate.	$V_n = V_0 R^n + \dfrac{d(R^n - 1)}{R - 1}$ $V_{60} = 255000 \times 1.006^{60} + \dfrac{1025(1.006^{60} - 1)}{1.006^{60} - 1}$ $= \$438869.90$
	3 Write a statement.	The lump sum available for retirement is $\$438\,869.90$.
c	**1** Write the perpetuity formula and list the values of V_0 and r.	**c** $d = \dfrac{V_0 r}{100}$ $V_0 = \$438\,869.90$ and $r = 8\%$ p.a.
	2 Substitute the values into the formula and find the value of the annual payment.	$d = \dfrac{438\,869.90 \times 8}{100}$ $= \$35\,109.59$ per year
	3 Calculate the monthly payment.	The monthly payment is $\dfrac{35\,109.59}{12} = \$2925.80$.
	4 Express the yearly payment as a percentage of his current annual salary.	$\dfrac{35\,109.59}{70\,000} \times 100\%$ $= 50.16\%$
	5 Write a statement.	The perpetuity will provide $\$2925.80$ a month which is equivalent to 50.16% of his current salary (in today's value of the money).
d	**1** State the values of V_0, A, n and R.	**d** $V_0 = \$438\,869.90, A = 0, n = 300(25 \times 12), R = 1.00\dot{6}$
	2 Write the annuities formula and substitute in the values.	$V_n = V_0 R^n - \dfrac{d(R^n - 1)}{R - 1}$ $0 = 438869.90 \times 1.006^{30} + \dfrac{d(1.00\dot{6}^{300} - 1)}{1.00\dot{6} - 1}$
	3 Calculate the monthly payment.	$d = \$3387.27$ The monthly payment is $\$3387.27$.
	4 Express the yearly payment as a percentage of his current annual salary.	The yearly payment will be $\$3387.27 \times 12 = \$40\,647.24$. As a percentage of his current annual salary, $\dfrac{40647.24}{70000} \times 100\% = 58.07\%$
	5 Write a statement.	The reducing balance annuity will provide $\$3387.27$ a month which is equivalent to 58.07% of his current salary (in today's value of the money).